

Finitary-Algebraic ‘Resolution’ of the Inner Schwarzschild Singularity

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A ‘resolution’ of the interior singularity of the spherically symmetric Schwarzschild solution of the Einstein equations for the gravitational field of a point-particle is carried out entirely and solely by finitistic and algebraic means. To this end, the background differential spacetime manifold and, *in extenso*, Differential Calculus-free purely algebraic (:sheaf-theoretic) conceptual and technical machinery of *Abstract Differential Geometry* (ADG) is employed. As in previous works [Mallios, A. and Raptis, I. (2001). Finitary spacetime sheaves of quantum causal sets: Curving quantum causality. *International Journal of Theoretical Physics*, **40**, 1885 [gr-qc/0102097]; Mallios, A. and Raptis, I. (2002). Finitary Čech-de Rham cohomology. *International Journal of Theoretical Physics*, **41**, 1857 [gr-qc/0110033]; Mallios, A. and Raptis, I. (2003). Finitary, causal and quantal vacuum Einstein gravity. *International Journal of Theoretical Physics* **42**, 1479 [gr-qc/0209048]], which this paper continues, the starting point for the present application of ADG is Sorkin’s finitary (:locally finite) poset (:partially ordered set) substitutes of continuous manifolds in their Gel’fand-dual picture in terms of discrete differential incidence algebras and the finitary spacetime sheaves thereof. It is shown that the Einstein equations hold not only at the finitary poset level of ‘discrete events,’ but also at a suitable ‘classical spacetime continuum limit’ of the said finitary sheaves and the associated differential triads that they define ADG-theoretically. The upshot of this is two-fold: On the one hand, the field equations are seen to hold when only finitely many events or ‘degrees of freedom’ of the gravitational field are involved, so that no infinity or uncontrollable divergence of the latter arises at all in our inherently finitistic-algebraic scenario. On the other hand, the law of gravity—still modelled in ADG by a *differential* equation proper—does *not* break down in any (differential geometric) sense in the vicinity of the locus of the point-mass as it is traditionally maintained in the usual manifold-based analysis of spacetime singularities in General Relativity (GR). At the end, some brief remarks are made on the potential import of ADG-theoretic ideas in developing a genuinely background-independent Quantum Gravity (QG). A brief

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comparison between the ‘resolution’ proposed here and a recent resolution of the inner Schwarzschild singularity by Loop QG means concludes the paper.

KEY WORDS: general relativity; C^∞ -smooth singularities; abstract differential geometry; sheaf theory; category theory; discrete differential incidence algebras of locally finite posets; causal sets; discrete Lorentzian quantum gravity.

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1. PROLEGOMENA: INTRODUCTION CUM PHYSICAL MOTIVATION

It is widely maintained today that, given certain broad assumptions and generic conditions, General Relativity (GR) ‘predicts’ singularities—*loci* in the spacetime continuum where the gravitational field grows uncontrollably without bound and, ultimately, the Einstein equations that it obeys ‘break down’ in one way or another. At the same time however, few physicists would disagree that the main culprit for these pathologies and their associated unphysical infinities is *our* model of spacetime as a pointed, C^∞ -smooth (:differential) manifold.³

Granted that the said anomalies and divergences are physically unacceptable, but at the same time that the whole conceptual edifice and technical machinery of Classical Differential Geometry (CDG)—the mathematical language in which GR is traditionally formulated⁴—vitaly depends on a base smooth manifold, the physicist appears to be impaled on the horns of a dilemma. On the one hand, she wishes to do away with singularities and their physically meaningless infinities, while on the other, she wishes to retain (or anyway, she is reluctant to readily abandon) the picture of a physical law (here, the law of gravity) as a *differential equation* proper, if anything in order for the theory still to be able to accommodate some notion of *locality*—be it *differential* locality.⁵

In other words, the tension may be expressed as follows: How can one get rid of the spacetime continuum with its ‘inherent’ singularities, but still be able to apply somehow *differential* geometric ideas to theoretical physics? Especially in GR, this friction manifests itself in the glaring conflict between the Principle of General Covariance (PGC) and the fruitless attempts so far at defining *precisely* what is a singularity in the theory (Geroch, 1968; Hawking and Ellis, 1973; Clarke, 1993; Rendall, 2005). For if one does away with the differential manifold

³ In the present paper, we tacitly identify the physicists’ intuitive term ‘*spacetime continuum*’ with the mathematicians’ notion of a (finite-dimensional) ‘*locally Euclidean space*’—i.e., a manifold, looking locally like \mathbb{R}^n and carrying the usual topological (C^0) and differential (C^∞) structure.

⁴ In the original formulation of GR by Einstein, CDG pertains to the pseudo-Riemannian geometry of smooth manifolds.

⁵ That is, the idea that dynamical gravitational field actions connect infinitesimally separated events, or equivalently, that events causally influence others in their ‘infinitesimal neighborhood’ (differential locality or local causality in the point-set manifold of events).

model for spacetime, and, as a result, the whole of the CDG based on it, one has also got to abandon the by now standard mathematical representation of the PGC by $\text{Diff}(M)$ —the diffeomorphism ‘symmetry’ group of automorphisms of the underlying smooth continuum M . No matter how easily the theoretical physicist may be convinced to abandon the *mathematical* (and quite *a priori!*) assumption of the spacetime continuum if the nonsensical singularities have to go with it, she will not be as easily prepared to sacrifice *the* pillar on which GR, as a *physical* theory, stands—the PGC. Otherwise, at least she is forced to look for an alternative mathematical expression for it—one that, unlike the traditional one involving $\text{Diff}(M)$, is *not* dictated by the smooth background manifold.

In order to appreciate how formidable this dilemma-*cum*-impasse is, one has to consider that, arguably, the only way we actually know how to do *differential* geometry is via a base manifold (i.e., CDG); albeit, in doing CDG we have to put up with the singularities that are built into M . Let it be stressed here that we tacitly assume that a *differential manifold* M is *nothing else but the algebra* $C^\infty(M)$ *of infinitely differentiable ‘coordinate’ functions labelling its points* (Gel’fand duality/spectral theory) (Mallios and Raptis, 2002, 2003, 2004). Thus, when we say that singularities are ‘inherent’ or built into M , we mean that they are singularities of some smooth function in $C^\infty(M)$.⁶

Of course, the theoretical physicist has time and again proven to be resourceful and inventive when confronted with such apparently insurmountable obstacles: For example, in a single stroke she may throw away the manifold picture of spacetime altogether⁷ and opt for a ‘discrete,’ finitistic model of spacetime and gravity. For, in any case, the general feeling nowadays is that very strong gravitational fields—probing smaller and smaller spacetime scales—such as those developing in the vicinity of a black hole, whose horizon is usually regarded as concealing a singularity in its core (e.g., the Schwarzschild black hole), only a quantum theory of gravity will be able to describe consistently (conceptually) and finitely (‘calculationally’).⁸ The implicit rationale (or at least the hope) here is that as the process of quantization (i.e., the development of QFT) somewhat alleviated the singularities and associated infinities of the classical field theories of matter (e.g., QED relative to classical Maxwellian electrodynamics), in the same way

⁶For example, the algebra of coordinates in which $g_{\mu\nu}$ —the principal dynamical variable in GR, whose components represent the 10 gravitational potentials—takes its values, is $C^\infty(M)$. That is, the said decade of potentials are *smooth* functions on M , and precisely because of this one says that the metric tensor $g_{\mu\nu}$ is a *smooth* field on M —an $\otimes_{C^\infty(M)}$ -tensor.

⁷Albeit, with a heavy heart, since if M has to go, so will CDG, so the continuous field theory based on it—a theory which has served her so well in the past: From the ever so successful (macroscopically) relativistic field theory of gravity (GR), to the equally if not more successful (microscopically) flat quantum field theories (QFT) of matter.

⁸Notwithstanding that singularities are normally regarded as a problem of GR *per se* (i.e., of *classical* gravity), long before quantization becomes an issue.

a quantization of GR may heal the singularities and related pathologies of the classical theory (even though the QF theoretic formalism still essentially relies on a background spacetime continuum, be it flat Minkowski space). There is also an even more ‘iconoclastic’ stance maintaining that both GR and quantum theory have to be modified somehow to achieve a fruitful union of the two into a consistent QG, which will then be able to shed more light on, if not resolve completely, the problem of singularities in GR (Penrose, 2003).

In other words, it is generally accepted today that GR appears to be out of its depth when trying to describe the gravitational field right at its source (e.g., the inner Schwarzschild singularity of the gravitational field of a point-particle (Finkelstein, 1958)), while at the same time, below the so-called Planck time-length—or equivalently, in dynamical processes (interactions) of very high energy-momentum transfer where quantum gravitational effects are supposed to become important—the classical spacetime continuum of macroscopic physics should give way to something more ‘reticular’ and ‘quantal.’

In summa, on the face of the aforementioned impasse and the subsequent hopes that QG could (or maybe, *should?*) remove singularities and their associated infinities in the end, *there* goes the spacetime manifold and, inevitably, *down* comes the whole CDG-edifice that is supported by it. The expression ‘*throw the baby away together with the bath-water*’ is perhaps suitable here, with the ever so valuable baby representing *differential geometric concepts and constructions*, while the epicurical aqueous ‘bathing medium’ standing for the ‘ambient’ base manifold, which apparently (but *only apparently*, as we will see in the sequel) vitally supports those CDG concepts and constructions. As a matter of fact, in the past, QG researchers have gone as far as to claim that

... at the Planck-length scale, differential geometry is simply incompatible with quantum theory ... [so that] one will not be able to use differential geometry in the true quantum-gravity theory ...” (Isham, 1991)

On the other hand, there is the recently developed *Abstract Differential Geometry* (ADG) (Mallios, 1998a,b, 2005b), a theory and method of doing differential geometry that does not employ at all a background geometrical C^∞ -smooth manifold, while at the same time it still retains, by using purely algebraic-categorical (:sheaf-theoretic) means, all the differential geometric conceptual panoply and technical machinery of the manifold-based CDG. This *differential geometric mechanism* of CDG, ADG has taught us both in theory and in numerous applications so far, is in essence of a purely algebraic character and quite independent of a base geometrical continuum, much in the relational way Leibniz had envisioned that Calculus should be formulated and practiced (Thompson, 2001). However, that ‘*fundamental algebraicity*’ is masked by the ‘geometric mantle’ of the background locally Euclidean space(time) M , which intervenes in our differential geometric calculations (i.e., in our Differential

Calculus!) in the guise of the smooth coordinates of (the points of) M in $C^\infty(M)$. Thus, *CDG* is a *background space(time)-dependent conception and method of differential geometry* that could be coined, in contradistinction to the base manifoldless Leibnizian ADG, Cartesian–Newtonian [In a significantly modified and expanded version of the e-arXiv posted paper; Mallios (2003, 2004b); Mallios and Raptis (2004)].

As a result, the relevance of ADG regarding the dilemma-*cum*-impasse posed above is that one not only is *not* forced to throw away the baby (:the differential mechanism) together with the bath-water (:the base manifold), but also that one can exercise that essentially algebraic differential geometric machinery in the very presence of singularities of any kind, literally as if the singularities were not there (Mallios and Rosinger, 1999; Mallios, 2001; Mallios and Rosinger, 2001, 2002; Mallios, 2002, 2003, 2005a; Mallios and Raptis, 2004). As it happens, ADG passes through the horns of the aforementioned dilemma by doing away with one horn (i.e., the base spacetime manifold), while showing at the same time that the gravitational field law—which is still algebraico-categorically represented by a base manifoldless version of the *differential* equations of Einstein—holds over, and by no means breaks down at, singularities of any sort.⁹ Consequently, the latter are not interpreted as being insuperable obstacles to, let alone break down points of, ‘differentiability’ as the manifold ‘mediated’ CDG (and consequently, the GR based on it) has so far (mis)led us to believe (Hawking and Ellis, 1973; Clarke, 1993; Mallios and Raptis, 2004).

In the present paper, we put ADG further to the test by applying it towards the ‘resolution’ (or better, as we shall see in the sequel, towards the total evasion or bypass) of the interior singularity of the spherically symmetric Schwarzschild solution of the (vacuum) Einstein equations for the gravitational field surrounding a point-particle of mass m . Classically (i.e., from the viewpoint of the manifold-based CDG and GR), this singularity, unlike the exterior one located at the so-called Schwarzschild black hole horizon-radius distance $r = 2m$ from the point-mass which has proven to be merely a ‘virtual,’ so-called *coordinate* one (Finkelstein, 1958), is thought of as being a ‘real,’ ‘genuine’ singularity as it resists any analytic (C^ω), smooth (C^∞), or even continuous (C^0), extension of the spacetime manifold M past it (Finkelstein, 1958; Hawking and Ellis, 1973; Clarke, 1993; Mallios and Raptis, 2004). In turn, the *differential* field equations of Einstein are thought of

⁹That is to say, not only in the presence of the usual, ‘classical’ as it were, singularities which are built into the smooth coordinates $C^\infty(M)$ of the pointed differential manifold M , but also with respect to more general, far more numerous and ‘robust’ ones, such as the so-called ‘*spacetime foam dense singularities*’ teeming Rosinger’s differential algebras of generalized functions (non-linear distributions). These are functions that are defined on finite-dimensional Euclidean and locally Euclidean (manifold) space(time)s, and include not only the smooth functions in $C^\infty(M)$, but also more general, ‘smearred out’ function(al)s, such as the linear distributions of Schwartz (Mallios and Rosinger, 1999, 2001; Mallios, 2001; Mallios and Rosinger, 2002; Mallios and Raptis, 2004).

as breaking down at the *locus* of the point-mass in the sense that they are no longer regarded as a valid description of gravitational dynamics right at the source of the gravitational field. As noted earlier, the general consensus nowadays is that only a QG will be able to describe gravitational dynamics for very strong, divergent from the pointed-continuum perspective and when treated with the usual analytic means of CDG (Calculus), gravitational fields near their massive (energy-momentum) sources. Even more dramatically and drastically, it is intuited that the said ‘infinistic’ manifold, by evoking a minimal, fundamental spacetime length-duration (ℓ_{p-t_p}), should be replaced by a ‘granular’ and ‘quantal’ structure that correctly represents the ‘true’ spacetime geometry in the quantum deep (Ashtekar, 2003).

In glaring contrast to the anticipations and hopes above, in this paper we will show by using the purely algebraic (:sheaf-theoretic), manifold- and *in extenso* Calculus-free ADG-theoretic means, that the (vacuum) Einstein equations not only do not break down in any sense, as *differential* equations proper, in the immediate vicinity of, or even right at, the Schwarzschild point-particle, but also that they hold both at the ‘discrete’ and at the ‘continuous’ background space(time) level of description of gravity. To this end, Sorkin’s finitary poset substitutes of continuous manifolds (Sorkin, 1991), in their Gel’fand-dual algebraic representation in terms of ‘discrete differential incidence algebras’ (Raptis and Zapatrin, 2000, 2001) and the finitary spacetime sheaves (finsheaves) thereof (Raptis, 2000b; Mallios and Raptis, 2001, 2002), are used *à la* ADG to show that the law of gravity (‘originating’ from the Schwarzschild point-mass) holds both at the ‘reticular-quantal’ level of description of spacetime (Mallios and Raptis, 2003) and in a (suitably defined) ‘classical,’ ‘continuum’ (inverse) limit of (a projective system of) the said finsheaves and the finitary differential triads that the latter comprise (Mallios and Raptis, 2002, 2003). We infer what has been already anticipated numerous times in the past trilogy (Mallios and Raptis, 2001, 2002, 2003) of applications of ADG to a (f)initary, (c)ausal, and (q)uantal (abbreviated ‘*fcq*’) version of Lorentzian gravity, namely, that *ADG allows us to develop a genuinely background spacetime-independent, purely gauge* (i.e., solely connection-based) *field theory of gravity*, no matter whether that base ‘spacetime’ is taken to be a continuum or a discretum.

Ex altis viewed, the paper is organized as follows: In the next section we review some ADG-basics from Mallios (1998a,b) that will prove to be useful in the sequel. In Section 3, we recall from the trilogy (Mallios and Raptis, 2001, 2002, 2003) the essentials from the ADG-theoretic approach, via Sorkin’s finitary discretizations of continuous manifolds (Sorkin, 1991), their Gel’fand-dual incidence algebraic representation (Raptis and Zapatrin, 2000, 2001) and the latter’s sheaf-theoretic picture (Raptis, 2000b), to a *fcq*-version of Lorentzian vacuum Einstein gravity. In Section 4, we bring forth from Papatriantafillou (2000, 2001, 2004, 2003, 2005), the key result from the categorical perspective on ADG, namely,

that the category of differential triads is bicomplete—i.e., closed under both projective and inductive limits. Having that result in hand, in the following Section (5) we present a direct ‘static’ (or ‘stationary’), ‘spatial’ (spacetime-localized) point-resolution of the interior Schwarzschild singularity and we anticipate an alternative ‘temporal’ (time-line extended), distributional one involving the so-called spacetime foam dense singularities from Mallios and Rosinger (2001, 2002). However, we leave the technical details of the latter for the more comprehensive treatment of C^∞ -gravitational singularities in Mallios and Raptis (2004). The paper concludes with a brief discussion on the possibility of developing a genuinely background-independent QG and we compare the Schwarzschild singularity resolution presented herein with similar recent efforts in the context of LQG (Modesto, 2004), passing at the same time the baton to Mallios and Raptis (2004) for a more thorough exposition of the potential import of ADG-ideas to current QG research.

2. RUDIMENTS OF ADG

We first recall from Mallios (1998a,b) some key concepts and structures in ADG that will prove to be useful in what follows.

2.1. \mathbf{K} -Algebraized Spaces

In ADG, we let X be an in principle *arbitrary topological space* on which a *sheaf* \mathbf{A} of unital, commutative, and associative \mathbb{K} -algebras A is erected. The coefficient field \mathbb{K} of the algebras may be taken to be either \mathbb{R} or \mathbb{C} . We tacitly assume that the *constant sheaf* $\mathbf{K} \equiv \mathbf{C}$ of \mathbb{K} -scalars is canonically embedded (injected) into \mathbf{A} : $\mathbf{K} \hookrightarrow \mathbf{A}$. We say that X is the *base space* (for the localization) of the *structure sheaf* \mathbf{A} of *generalized arithmetics*.¹⁰ The pair

$$\mathbf{D} := (X, \mathbf{A}_X) \tag{1}$$

is called a \mathbf{K} -*algebraized space*.

2.2. Vector Sheaves and Differential Triads

Technically speaking, by a *vector sheaf* \mathcal{E} in ADG we mean a *locally free \mathbf{A} -module of finite rank*, that is to say, a sheaf of \mathbf{A} -modules over X that is locally expressible as a finite power (a finite Whitney sum) of \mathbf{A}

$$\mathcal{E}|_U \simeq \mathbf{A}^n|_U = (\mathbf{A}|_U)^n = \mathbf{A}^n(U) = \mathbf{A}(U)^n \quad (U \text{ open in } X) \tag{2}$$

with $\mathbf{A}^n(U) = \mathbf{A}(U)^n := \Gamma(U, \mathbf{A})$ the local sections of \mathbf{A} .

¹⁰The terms ‘*coefficients*’ or ‘*coordinates*’ will be used interchangeably with the term ‘*arithmetics*’ in the sequel.

We also assume that *the dual of \mathcal{E}*

$$\mathcal{E}^* := \Omega(\equiv \Omega^1) = \mathcal{H}om_{\mathbf{A}}(\mathcal{E}, \mathbf{A}) \quad (3)$$

is the ADG-theoretic analogue of the sheaf of modules of smooth 1-forms in the classical, manifold-based theory (CDG). It must be emphasized here that CDG is ‘recovered’ from ADG when one assumes \mathcal{C}_X^∞ for structure sheaf \mathbf{A} of coordinates in the theory, which in turn means that X is a smooth manifold (Gel’fand duality and spectral theory).

Now, having defined \mathbf{D} s, \mathcal{E} s and their duals Ω , we are in a position to define *the fundamental notion in ADG, that of a differential triad \mathfrak{T}* . It is a triplet

$$\mathfrak{T} := (\mathbf{A}_X, \partial, \Omega_X^1) \quad (4)$$

consisting of a structure sheaf \mathbf{A}_X on some topological space X (i.e., a \mathbf{K} -algebraized space \mathbf{D} is built into every \mathfrak{T})¹¹ and a \mathbf{K} -linear Leibnizian sheaf morphism ∂ . That is to say, ∂ is a map

$$\partial : \mathbf{A} \longrightarrow \Omega^1 \quad (5)$$

which is \mathbf{K} -linear, and for every two local sections p and q in $\Gamma(U, \mathbf{A}) \equiv \mathbf{A}(U)$ (:the collection of local sections of \mathbf{A} over $U \subset X$), the usual Leibniz rule is observed

$$\partial(p \cdot q) = p \cdot \partial(q) + q \cdot \partial(p) \quad (6)$$

2.3. \mathbf{A} -Connections

The basic observation of ADG is that *the basic differential operator ∂ in differential geometry is the archetypical instance of an \mathbf{A} -connection*¹²—albeit, a flat connection as we shall see later.¹³ Thus, a general (curved) \mathbf{A} -connection \mathcal{D} in ADG is an abstraction from and a generalization of the usual ∂ , defined as the following \mathbf{K} -linear sheaf morphism

$$\mathcal{D} : \mathcal{E} \longrightarrow \Omega(\mathcal{E}) \equiv \mathcal{E} \otimes_{\mathbf{A}} \Omega \cong \Omega \otimes_{\mathbf{A}} \mathcal{E} \quad (7)$$

¹¹ For typographical economy, from now on we will omit the base space X as a subscript to the sheaves involved.

¹² In ADG, the concept of an algebraic \mathbf{A} -connection is *the fundamental one*, about which the whole theory revolves. \mathbf{A} -connections are the *raison d’être* of ADG (Mallios, 1988, 1998a,b).

¹³ Moreover, in complete analogy to ∂ , one can then iteratively define higher order prolongations d^i ($i \geq 1$) of $\partial \equiv d^0$, which again are \mathbf{K} -linear and Leibnizian sheaf morphisms between \mathbf{A} -modules Ω^i of differential form-like entities of higher degree $d^i : \Omega^i \longrightarrow \Omega^{i+1}$ ($\mathbf{A} \equiv \Omega^0$), satisfying at the same time the usual nilpotency condition of the standard (exterior Cartan-de Rham-Kähler) differential operator d : $d^{i+1} \circ d^i \equiv d^2 = 0$ (with d^2 being ‘the square of d ,’ not to be confused with the second-order prolongation of ∂).

2.4. Curvature of A-Connections

With \mathcal{D} at our disposal, we can define its curvature $R(\mathcal{D})$ diagrammatically as follows

$$\begin{array}{ccc}
 \mathcal{E} & \xrightarrow{\mathcal{D}} & \Omega^1(\mathcal{E}) \equiv \mathcal{E} \otimes_{\mathbf{A}} \Omega^1 \\
 & \searrow R \equiv \mathcal{D}^1 \circ \mathcal{D} & \swarrow \mathcal{D}^1 \\
 & & \Omega^2(\mathcal{E}) \equiv \mathcal{E} \otimes_{\mathbf{A}} \Omega^2
 \end{array} \tag{8}$$

for a suitable higher order extension \mathcal{D}^2 of $\mathcal{D}(\equiv \mathcal{D}^1)$.¹⁴ It must be noted here that, from the definition of $R(\mathcal{D})$ above, it follows that the nilpotent ∂ is a *flat* connection—i.e., $R(\partial) := d^2 = 0$. It is also important to observe here that, unlike \mathcal{D} which is only a \mathbf{K} -sheaf morphism, its curvature $R(\mathcal{D})$ is an \mathbf{A} -morphism, that is to say, our generalized arithmetics (coordinates) in \mathbf{A} respect it. Equivalently, and philologically speaking, R ‘sees through’ our generalized arithmetics (coordinates) in \mathbf{A} . On the other hand, our acts of coordinatization in \mathbf{A} cannot ‘capture’ \mathcal{D} , which eludes them since it is not an \mathbf{A} -morphism (Mallios, 1998a; Mallios and Raptis, 2002, 2003).¹⁵

2.5. Manifoldless (Pseudo)-Riemannian Geometry and Vacuum Einstein Equations in a Nutshell

Following Mallios (1998a,b, 2001); Mallios and Raptis (2001, 2002, 2003), we can then formulate ADG-theoretically, *in a manifestly background (spacetime) manifoldless way*, all the concepts and structures of the CDG-based (pseudo)-Riemannian geometry underlying GR such as \mathbf{A} -valued (Lorentzian) metrics ρ , Christoffel \mathbf{A} -connections ∇ compatible with ρ (i.e., metric or torsionless connections), the Ricci curvature $\text{End}\mathcal{E}$ -valued¹⁶ $\otimes_{\mathbf{A}}$ -tensor \mathcal{R} , and its \mathbf{A} -valued trace-contraction—the Ricci scalar \mathcal{R} .

The upshot of our brief *résumé* here of the application of ADG to GR isxxxxsxsx that the vacuum Einstein equations read in our scheme

$$\mathcal{R}(\mathcal{E}) = 0 \tag{9}$$

¹⁴Like the higher order extensions d^i of $\partial \equiv d^1$ mentioned in the last footnote, \mathcal{D}^2 for example is a \mathbf{K} -linear, Leibnizian sheaf morphism between Ω^1 and Ω^2 : $\mathcal{D}^2 : \Omega^1(\mathcal{E}) \longrightarrow \Omega^2(\mathcal{E})$. It acts locally (i.e., section-wise), and relative to \mathcal{D} , as follows: $\mathcal{D}^2(p \otimes q) := p \otimes dq - q \wedge \mathcal{D}q$, ($p \in \mathcal{E}(U)$, $q \in \Omega^1(U)$, U open in X).

¹⁵This observation about $R(\mathcal{D})$ will become important when we discuss the \mathbf{A} -functoriality of the ADG-theoretic formulation of (vacuum) gravitational dynamics in (9) later.

¹⁶And recall that, locally: $\text{End}\mathcal{E}(U) = M_n(\mathbf{A})(U)$.

recalling at the same time from Mallios (2001); Mallios and Raptis (2003, 2004) a couple of important observations pertaining to them:

1. *From the ADG-theoretic viewpoint, GR is another gauge theory*—in fact, a ‘*pure gauge theory*’ as the only dynamical variable involved is the (curvature of the) gravitational \mathbf{A} -connection \mathcal{D} , and no external smooth base spacetime manifold is employed. This is in glaring contrast to *both* the original (smooth) spacetime metric-based formulation of GR by Einstein (second-order formalism) *and* to the recent ‘new variables’ formulation of GR by Ashtekar (Ashtekar, 1986) (first-order formalism reminiscent of Palatini’s metric-affine one) which, although it emphasizes the importance of the notion of connection so as to place gravity in the category of gauge forces, it still employs a smooth background manifold, while at the same time the (smooth) metric of the second-order formalism is still present ‘in disguise,’ being encoded in the (smooth) *vierbein* field variables. Due to these features, we coin the ADG-formulation of GR ‘*half-order, pure gauge formalism*.’¹⁷
2. The vacuum Einstein equations derive variationally (solely with respect to \mathcal{D} !) from the ADG-theoretic version of the Einstein–Hilbert action functional $\mathfrak{E}\mathfrak{H}$, which is an \mathbf{A} -valued functional on the affine space $\mathbf{A}_{\mathbf{A}}(\mathcal{E})$ of the \mathbf{A} -connections \mathcal{D} , which in turn becomes the relevant configuration space in our theory of gravity.
3. Since there is no external smooth spacetime continuum involved in the ADG-version of GR, the principle of general covariance (PGC) of the usual manifold-based theory is not expressed via $\text{Diff}(M)$ as usual, but via $\text{Aut}\mathcal{E}$ —the (group sheaf of) automorphisms (dynamical self-transmutations) of the gravitational field itself. Here one might wish to recall that in ADG the term *field* pertains to the pair $(\mathcal{E}, \mathcal{D})$, with \mathcal{E} the (geometric) representation (or carrier) space of the (algebraic) connection field \mathcal{D} (Mallios, 1998a,b; Mallios and Raptis, 2003, 2004). In technical jargon, \mathcal{E} is the associated (representation) sheaf of the principal sheaf $\text{Aut}\mathcal{E}$ of field automorphisms (Vassiliou, 1994, 1999, 2000, 2003). Moreover, since \mathcal{E} is by definition locally isomorphic to \mathbf{A}^n , $\text{Aut}\mathcal{E}(U) := (\text{End}\mathcal{E}(U))^\bullet \equiv (M_n(\mathbf{A}))^\bullet$. This is an autonomous conception of covariance, pertaining directly to the gravitational field ‘in-itself.’

¹⁷ ‘*Half-order*,’ because only \mathcal{D} , and not $g_{\mu\nu}$ (second-order) or e_μ^a (first-order), is engaged in the dynamics (and in the first-order formalism there are *two* basic variables engaged in the dynamics: The \mathcal{C}^∞ -connections and the smooth comoving frame-tetrads). ‘*Pure gauge*,’ because there is no ‘external’ spacetime (manifold) involved—only ‘internal,’ gauge ‘degrees of freedom’ associated with the gravitational connection field \mathcal{D} ‘in-itself.’ In the concluding section, we will return to comment further on this virtue of the ADG-formulation of gravity and its implications for developing a genuinely background-independent QG.

without reference to an external spacetime manifold, which we have elsewhere coined ‘*synvariance*.’

In connection with our remarks above about gravity as a ‘pure gauge theory’ *à la* ADG, no *external* spacetime (manifold) symmetries in the guise of $\text{Diff}(M)$ appear in our theory—only the ‘*internal*,’ gauge ones $\text{Aut}\mathcal{E}$ of the field $(\mathcal{E}, \mathcal{D})$ ‘in-itself’ are involved. In fact, the distinction external/internal symmetries loses its meaning in our ADG-perspective on gravity. Of course, assuming C_X^∞ for structure sheaf—i.e., that X is a differential manifold M —one may recover, if one wishes, the external $\text{Diff}(M)$ used in the mathematical expression of the PGC of the CDG and smooth manifold-based GR since, by definition, $\text{Aut}M \equiv \text{Diff}(M)$. It also follows now that the relevant configuration space is the aforementioned affine space $A_A(\mathcal{E})$ of \mathbf{A} -connections modulo the field’s dynamical self-transmutations (‘autosymmetries’) in $\text{Aut}\mathcal{E}: A_A(\mathcal{E})/\text{Aut}\mathcal{E}$.

4. Finally, closely related to the remarks about synvariance above is the issue of *functoriality*. In the ADG perspective on GR, functoriality pertains to the fact that the gravitational dynamics—the vacuum Einstein equations (9)—is expressed via the curvature of the connection, which is an \mathbf{A} -morphism—or equivalently, a $\otimes_{\mathbf{A}}$ -tensor ($\otimes_{\mathbf{A}}$ being the homological tensor product functor). This means that *the generalized coordinates in \mathbf{A} , that we employ in order to ‘measure’ or ‘geometrically represent’ (and ‘localize’ in \mathcal{E} over X) the gravitational connection field \mathcal{D} , respect it.*¹⁸ Moreover, since if any space(time) is involved at all in our scheme, then it is regarded as being built into the \mathbf{A} that we assume in the first place to coordinatize (or geometrically represent) the gravitational connection field \mathcal{D} (on \mathcal{E}),¹⁹ the gravitational dynamics, being \mathbf{A} -functorial, ‘sees through’ the said ‘*spectral space(time)*’ inherent in \mathbf{A} .

Precisely in this \mathbf{A} -functoriality lies the strength and import of ADG *vis-à-vis* (gravitational) singularities, in the sense that one can ‘*absorb*,’ ‘*incorporate*, or ‘*integrate* into \mathbf{A} singularities of any kind—ones that are arguably more robust and numerous than the C^∞ -ones built into the usual coordinate structure sheaf C_M^∞ of the smooth manifold—and still be able to

¹⁸ Although it must be stressed here that the connection itself, being simply a \mathbf{K} -morphism, is *not* an \mathbf{A} -morphism or $\otimes_{\mathbf{A}}$ -tensor, thus it ‘eludes’ our measurements in \mathbf{A} . However, it is the curvature of the connection that appears in (9), which *is* an \mathbf{A} -morphism. \mathcal{D} is a purely algebraic notion, and as such it evades our generalized acts of measurement or ‘geometrization’ (and concomitant representation on the associated sheaf \mathcal{E}) of the gravitational field \mathcal{D} , which are organized in \mathbf{A} (Mallios and Raptis, 2003, 2004).

¹⁹ What we have in mind here is a generalized version of the notion of Gel’fand duality whereby, in the same way that in the classical theory (CDG) one obtains a smooth manifold M as the Gel’fand spectrum of topological algebra $C^\infty(M)$ (or equivalently, from $\mathbf{A} \equiv C_M^\infty$) (Mallios, 1986, 1993), one can (spectrally) extract other ‘geometrical’ base space(time)s from various different choices of structure algebra sheaves \mathbf{A} (indeed, by assuming ‘functional’ structure sheaves other than C_M^∞).

show that the gravitational field equations hold and in no way break down at their *loci* in X . As it were, *the differential equations of Einstein hold over and above them, in spite of their presence in the \mathbf{A} being employed* (Mallios and Rosinger, 1999, 2001; Mallios, 2001, 2002; Mallios and Rosinger, 2002; Mallios, 2005a; Mallios and Raptis, 2004).

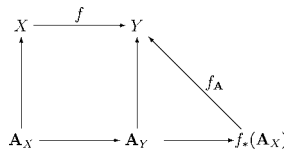
2.6. The Categorical Imperative of ADG

Throughout the present paper we have mentioned various category-theoretic sounding terms, as for example the notions of *sheaf morphism* and *functoriality*. Indeed, on the whole one can say that ADG is an algebraico-categorical scheme for doing differential geometry (Mallios, 1998a,b), for after all, “*the methods of sheaf theory are algebraic*” (Grauert and Remmert, 1984). Here, we expose briefly some key categorical aspects of ADG as explored in great depth in Papatriantafillou (2000, 2001, 2004, 2003, 2005).

The first thing to mention is that one can regard differential triads as objects in a category \mathfrak{DT} —the *category of differential triads* (Papatriantafillou, 2000, 2005). The arrows in \mathfrak{DT} are *triad morphisms*, whose definition we now readily recall from Papatriantafillou (2000, 2003, 2005)

One lets X and Y be topological spaces, assumed to be the base spaces of some \mathbf{K} -algebraized spaces (X, \mathbf{A}_X) and (Y, \mathbf{A}_Y) , respectively. In addition, one lets $\mathfrak{T}_X = (\mathbf{A}_X, \partial_X, \mathbf{\Omega}_X)$ and $\mathfrak{T}_Y = (\mathbf{A}_Y, \partial_Y, \mathbf{\Omega}_Y)$ be differential triads over them. Then, a *morphism \mathcal{F} between \mathfrak{T}_X and \mathfrak{T}_Y is a triplet of maps $\mathcal{F} = (f, f_A, f_{\mathbf{\Omega}})$* , enjoying the following four properties:

1. The map $f : X \longrightarrow Y$ is *continuous*;
2. The map $f_A : \mathbf{A}_Y \longrightarrow f_*(\mathbf{A}_X)$ is a *morphism of sheaves of \mathbb{K} -algebras over Y* preserving the respective algebras’ unit elements (i.e., $f_A(1) = 1$);²⁰ and the following categorical diagram is obeyed:



²⁰ In the expression for f_A above, f_* is the *push-out* along the continuous f , a map which carries each element of a differential triad into a like element in the sense that, for any triad \mathfrak{T} , $f_*(\mathfrak{T}) := (f_*(\mathbf{A}), f_*(\partial), f_*(\mathbf{\Omega}))$ is also a differential triad—the one ‘induced’ by f (Papatriantafillou, 2003, 2004); whence, term-wise for our triads \mathfrak{T}_X and \mathfrak{T}_Y above (and omitting the base topological space subscripts): $f_*(\mathbf{A}) := (f_*(\mathbf{A})(U) := \mathbf{A}_{f^{-1}(U)})$, ($U \subseteq Y$ open) is a sheaf of unital, abelian, associative \mathbb{K} -algebras over Y , $f_*(\mathbf{\Omega}) := (f_*(\mathbf{\Omega})(U) := \mathbf{\Omega}_{f^{-1}(U)})$, ($U \subseteq Y$ open) a sheaf of $f_*(\mathbf{A})$ -modules (of first-order differential form-like entities), and $f_*(\partial) := (f_*(\partial)(U) := \partial_{f^{-1}(U)})$, ($U \subseteq Y$ open) an induced \mathbf{K} -linear, Leibnizian sheaf morphism (Papatriantafillou, 2003).

3. The map $f_\Omega : \Omega_Y \longrightarrow f_*(\Omega_X)$, as noted in the last footnote, is a *morphism of sheaves of \mathbb{K} -vector spaces over Y* , with $f_\Omega(\alpha\omega) = f_A(\alpha)f_\Omega(\omega)$, $\forall \alpha \in \mathbf{A}_Y$, $\omega \in \Omega_Y$; and finally,
4. with respect to the $\mathbf{C} \equiv \mathbf{K}$ -sheaf morphism (*viz.* flat connection) ∂ in the respective triads, and as it has also been alluded to in the last footnote, the following diagram is commutative:

$$\begin{array}{ccc}
 \mathbf{A}_Y & \xrightarrow{\partial_Y} & \Omega_Y \\
 f_A \downarrow & & \downarrow f_\Omega \\
 f_*(\mathbf{A}_X) & \xrightarrow{f_*(\partial_X)} & f_*(\Omega_X)
 \end{array}$$

which reads: $f_\Omega \circ \partial_Y = f_*(\partial_X) \circ f_A$.

In summa, \mathcal{DT} is a category having $\mathcal{T}s$ for objects and $\mathcal{F}s$ for arrows. Let it be noted here that in the past it has been amply observed that differential triads are generalizations of differential manifolds. Indeed, the entire differential structure of a \mathcal{C}^∞ -smooth manifold M is encoded in the *classical differential triad* \mathcal{T}_∞ having as \mathbf{A} the sheaf of germs of local ($\mathbb{K} \equiv \mathbb{R}$ -valued) \mathcal{C}^∞ -functions on M , as Ω the usual sheaf of germs of local \mathcal{C}^∞ -differential 1-forms (i.e., $\Omega \equiv \Gamma^\infty(T^*M)$), and one can identify ∂ with the usual (exterior) derivative d : $\partial \equiv d : \mathbf{A} \longrightarrow \Omega : \alpha \in \mathbf{A} \mapsto \partial(\alpha) := d\alpha \in \Omega$. It must be also stressed that \mathcal{T}_∞ is only a particular instance of the general (abstract notion of) differential triad, which, as noted earlier, is able to accommodate algebraized spaces (and differentials ∂ on them) other than the classical one $\mathbf{D}_\infty = (M, \mathcal{C}_M^\infty)$ (and $\partial \equiv d$)—i.e., algebraized spaces hosting structure sheaves other than \mathcal{C}_X^∞ , and possibly non-functional (of course, as long as these generalized arithmetics provide one with the fertile ground on which to define a ∂ or \mathcal{D} *à la* (5) or (7), and thus to develop differential geometric ideas with them).

But let us discuss a bit more this categorical versatility of the differential triads of ADG compared to the ‘rigidity’ and associated shortcomings of (the category of) smooth manifolds.

2.7. Brief Discussion of the Categorical Versatility of ADG

The categorical ‘versatility’ and ‘flexibility’ of ADG, compared to the ‘crystalline rigidity’ of the manifold-based CDG, may be summarized by outlining the following shortcomings of Man —the category of (finite-dimensional) differential (\mathcal{C}^∞ -smooth) manifolds—relative to \mathcal{DT} :

1. *Man* has no initial or final structures. That is, one cannot pull-back or push-out a smooth atlas by a continuous map.

2. *The quotient space of a manifold by an (arbitrary) equivalence relation is not a manifold.*
3. *Similarly, an arbitrary subset of a manifold is not a manifold. In other words, $\mathcal{M}an$ has no canonical subobjects.*
4. *In general, $\mathcal{M}an$ is not closed under inductive (direct) or projective (inverse) limits.²¹ Another way to say this is that $\mathcal{M}an$ is not bicomplete (i.e., complete and co-complete).*
5. *Generally, there are no well defined categorical products and co-products in $\mathcal{M}an$.*

As Papatriantafillou has shown in a long series of thorough investigations (Papatriantafillou, 2000, 2001, 2004, 2003, 2005), \mathcal{DT} not only does not suffer from such (differential geometric) maladies, but also goes all the way towards healing or bypassing them completely. Thus, from a mathematical point of view alone, theoretical physics' applications aside, these differential geometric anomalies of $\mathcal{M}an$ could suffice for motivating the development of ADG—in fact, they could be regarded as the *raison d'être et de faire* of ADG. In particular, and of special importance to the present paper as we shall see in the sequel, Papatriantafillou has shown in connection with the differential manifolds' deficiencies 1, 2, and 4 above, that in \mathcal{DT} :

- And we quote, “*the differential mechanism induced by a differential triad is transferred backwards and forward by any continuous map f . The initial and final structures thus obtained satisfy appropriate universal conditions that turn the continuous map f into a differentiable map.*” (Papatriantafillou, 2004, 2003). To recapitulate in a nutshell this result, given a continuous map $f : X \rightarrow Y$, with X the base space of a differential triad \mathfrak{T}_X , Papatriantafillou showed that f pushes forward the (essentially algebraic) differential mechanism of \mathfrak{T}_X , so that a new and unique differential triad—one that satisfies a *universal mapping* condition (Papatriantafillou, 2003)—is defined on Y , so that in the process, f becomes differentiable. The relevant theorem,²² which uses some ideas already mentioned *en passant* in footnote 18 before, can be stated as follows:²³

Theorem: Let $\mathfrak{T}_X = (\mathbf{A}_X, \partial_X, \mathbf{\Omega}_X) \in \mathcal{DT}_X$,²⁴ and $f : X \rightarrow Y$ continuous. When Y inherits $f_*(\mathfrak{T}_X) := (f_*(\mathbf{A}_X), f_*(\partial_X), f_*(\mathbf{\Omega}_X))$ from the push-out f_* of f , then there is a morphism of differential triads $\mathcal{F} =$

²¹ In category-theoretic jargon, projective (inverse) limits are known as ‘*categorical limits*,’ while inductive (direct) ones as ‘*categorical colimits*.’

²² Theorem in Papatriantafillou (2004).

²³ For the corresponding detailed proof, the reader is referred to Papatriantafillou (2004).

²⁴ Plainly, \mathcal{DT}_X is the subcategory of \mathcal{DT} consisting of all differential triads and triad morphisms with common base topological space X .

$(f, f_A, f_\Omega) : \mathfrak{T}_X \longrightarrow f_*(\mathfrak{T}_X)$ ($\in \mathfrak{DT}$)—i.e., f becomes *differentiable*. Moreover, the pushed-forward triad $f_*(\mathfrak{T}_X)$ satisfies the following universal (composition) property (Papatriantafillou, 2003): given a triad $\mathfrak{T}_Y = (A_Y, \partial_Y, \Omega_Y) \in \mathfrak{DT}_Y$, and a morphism $\tilde{\mathcal{F}} := (f, \tilde{f}_A, \tilde{f}_\Omega) : \mathfrak{T}_X \longrightarrow \mathfrak{T}_Y$, there is a unique morphism $(id_Y, g_A, g_\Omega) : f_*(\mathfrak{T}_X) \longrightarrow \mathfrak{T}_Y$ such that $\tilde{\mathcal{F}} = (id_Y, g_A, g_\Omega) \circ \mathcal{F}$.

Accordingly, the ‘dual’ (converse) scenario involving f ’s *pull-back action* f^* , when now the range of f is a differential triad \mathfrak{T}_Y on Y while X (f ’s domain) is simply a topological space not being endowed *a priori* with a differential (triad) structure, f^* too can be seen to transfer (induce) on X the differential mechanism encoded in \mathfrak{T}_Y , thus rendering X a *differential* (not just a topological) space and in the process promoting f to a *differentiable* (not just a continuous) map (Papatriantafillou, 2004).

Of great mathematical interest is that these results may serve as the starting point for research into what one might call the ‘*differential geometry of topological spaces*,’ and they depict some sort of ‘Calculus-reversal,’ since in the usual theory, ‘*differentiability implies continuity*,’ while here in some sense ‘*continuity* (i.e., topology, plus algebraic structure—e.g., the employment of a topological vector space structure) *entails differentiability*.’ Indeed, *differentiability* (i.e., the ability to define a derivative/differential operator) *is a topologico-algebraic notion*—one that is secured in the manifold-based CDG exactly because $\mathcal{C}^\infty(M)$ is a (non-normable) topological algebra (Mallios and Raptis, 2003, 2004).

- When a manifold M is factored by an equivalence relation \sim , and there happens to be a continuous map f from M to the resulting quotient space $\tilde{M} = M/\sim$ (suitably topologized), then the result in 1 above secures that the classical differential structure (i.e., differential triad) on M can be pushed-forward by f_* on the ‘moduli space’ \tilde{M} , thus endow it with a differential triad of its own. In the next section, we will encounter a concrete example of this ‘*differential triad induction from a continuum to a discretum*’ having to do with Sorkin’s finitary T_0 -poset discretizations of continuous (\mathcal{C}^0) manifold topologies (Sorkin, 1991).
- Finally, as Papatriantafillou (2001) has shown and in the forthcoming monograph (Papatriantafillou, 2005), \mathfrak{DT} , unlike *Man*, is bicomplete—that is to say, it is closed under projective and inductive limits. This virtue of \mathfrak{DT} will prove to be of paramount importance on the one hand in Section 4, where we give the ‘classical continuum limit’ of *fcq*-differential triads and of the *fcq*-version of the vacuum Einstein equations (9) holding on them, and on the other, in Section 5, where we provide an explicit, ‘constructive’ evasion of the interior Schwarzschild singularity by finitistic-algebraic means as already developed under the prism of ADG (and briefly

summarized in the next section) in the past tetralogy (Mallios and Raptis, 2001, 2002, 2003, 2004).

3. APPLICATION OF ADG TO FINITARY, CAUSAL, AND QUANTAL LORENTZIAN GRAVITY

For expository completeness, let us first recall from the trilogy (Mallios and Raptis, 2001, 2002, 2003) the basic results and constructions that led us to formulate an *fcq*-version of vacuum Einstein–Lorentzian gravity with the help of ADG as these will be used in Section 5 to achieve our main goal here, namely, to evade the inner Schwarzschild singularity purely finitistically and algebraically, and in a ‘constructive’ fashion.

3.1. Sorkin’s Finitary Substitutes of Continuous Manifolds: Topology (‘Continuity’) from Order

A brief history of *fcq*-vacuum Einstein gravity begins with Sorkin’s finitary poset discretizations of continuous (i.e., topological, otherwise known as \mathcal{C}^0 -) manifolds.

The original idea in Sorkin (1991) is, starting with an open bounded region X^{25} in a manifold M^{26} , to cover it with a locally finite open covering \mathcal{U}_i . One may recall that a cover $gauge_i$ of X is called locally finite whenever every point of X has an open neighborhood about it that meets a finite number of the covering sets. The index ‘ i ’ of the open covering will be explained shortly. Then, it was observed that X can be replaced by a ‘discrete’²⁷ T_0 -topological space P_i , having the structure of a poset, when the following equivalence binary relation \sim relative to \mathcal{U}_i is imposed on its points:

$$\forall x, y \in X : x \overset{\mathcal{U}_i}{\sim} y \Leftrightarrow \Lambda(x)|_{\mathcal{U}_i} = \Lambda(y)|_{\mathcal{U}_i} \quad (10)$$

$$\Lambda(x)|_{\mathcal{U}_i} := \bigcap \{U \in \mathcal{U}_i : x \in U\}$$

where, clearly, $\Lambda(x)|_{\mathcal{U}_i}$ is the ‘smallest’ open set in \mathcal{U}_i containing x , which we here coin ‘*Sorkin’s ur-cell*’ (relative to \mathcal{U}_i).

²⁵ By ‘bounded’ it is meant that X ’s closure is compact, a space otherwise known as *relatively compact*.

²⁶ Let it be stressed here that Sorkin was interested only in the standard continuous (\mathcal{C}^0 -) topology of M and no allusion to its differential (smooth) structure was made. Also, there is no harm in assuming the usual Hausdorff (T_2) topology for M , although Sorkin’s results follow even from a weaker T_1 assumption.

²⁷ From now on we will often put ‘discrete’ in single quotation marks so that one does not confuse it with the technical term ‘discrete topological space’ referring to the (trivially Hausdorff) topology of a totally disconnected set, all the points of which are ‘clopen’ (i.e., closed and open). Even when these quotation marks are omitted, we do *not* mean the set with completely disjoint points, unless specifically noted.

The aforementioned T_0 -poset P_i , called ‘the finitary substitute of the continuous topology of X ,’ is then obtained as the quotient of X by \sim^{U_i} :

$$P_i = X / \sim^{U_i} \tag{11}$$

Plainly, the elements of P_i are \sim^{U_i} -equivalence classes of X ’s points, with the equivalence relation being interpreted as ‘indistinguishability’ or ‘non-separability’ of X ’s points by the covering sets of U_i . In other words, the ‘points’ of P_i are Sorkin’s *ur*-cells $\Lambda(x)|_{U_i}$ while the points of the original space(time) X have been substituted, ‘blown up,’ or even ‘smeared’ so to speak, by ‘larger’ open sets about them. Sorkin initially appreciated that operationally realistic determinations (‘measurements’) of space(time) locution can be modelled after coarse regions in the said space(time), while the continuum, the ‘sharp’ points of which “*carrying its continuous topology*” (Sorkin, 1991), is an ideal theoretical construct not corresponding to “*what we actually do to produce spacetime by our measurements*” (Sorkin, 1995). Let us note *en passant*, the said ‘operational pragmatism’ aside, that it is widely recognized today that the pathologies of the spacetime continuum (e.g., the singularities of GR, or even the unphysical infinities of QFT) are mainly due to its ideal, point-like character, or equivalently, of the ideal point-like ‘nature’ of the matter sources (:particles) of the fields involved. Arguably, quantum (field) theory goes some way towards alleviating the infinities assailing its classical counterpart exactly because it sets a fundamental limit (a regularization cut-off scale of the order of Planck) to the ideal assumption in the classical (field) theory of infinite (spacetime) point-localization of the relevant fields, which in turn in the quantum theory are usually modelled after ‘smeared’ and ‘blown-up’ (operator-valued) distributions.

The important interpretation of the P_i s in Sorkin (1991) as discrete approximations of the topological manifold X comes from considering an *inverse system* (or net) $\overline{P} = \{P_i\}$ of such finitary substitutes, and of continuous surjection maps f_{ji} between them, in the sense that

$$P_i \leq P_j \Leftrightarrow P_j \xrightarrow{f_{ji}} P_i \tag{12}$$

where \leq is the act of topological refinement of the P_i s²⁸ corresponding to the employment of more numerous and ‘smaller’ open sets (i.e., finer-and-finer U_i s) to cover X ’s points.

²⁸Roughly, the partial order $P_i \leq P_j$, which comes from the partial ordering of the U_i s in an i -indexing net thereof and reading ‘*the open covering U_j is finer than U_i* ’ (or equivalently, that the subtopology τ_i of X generated by finite intersections of arbitrary unions of the U_i s in U_i is included in the corresponding τ_j : $\tau_i \subseteq \tau_j$ —*alias*, τ_i is coarser than τ_j), means that there is a continuous surjection f_{ji} from the topological T_0 -poset P_j to P_i . The epithet ‘continuous’ for f_{ji} pertains to the fact that one can assign a ‘natural’ topology—the so-called Sorkin lower-set topology—to the P_i s, whereby an open set is of the form $\mathcal{O}(x) := \{y \in P_i : y \longrightarrow x\}$, and where \longrightarrow is the partial order relation in P_i (with basic open sets involving the links or covering—‘immediate arrow’—relations in P_i). Plainly then, f_{ji} is a monotone (partial order-preserving) surjection from P_j to P_i , hence continuous with respect to the Sorkin topology.

Now, the central result in Sorkin (1991), one that qualifies the P_i s as genuine topological approximations of the continuum X , is that *the said inverse (projective) system $\overleftarrow{\mathcal{P}}$ possesses an inverse (projective) limit space—call it $P_\infty = \lim_{\leftarrow} \overleftarrow{\mathcal{P}}$ —that is practically homeomorphic to the original \mathcal{C}^0 -manifold that we started with.*²⁹ The physical interpretation of the inverse limit procedure is that as one employs finer-and-finer open sets to cover X 's points, at the limit of infinite refinement of the corresponding \mathcal{U}_i s, one obtains a space that is essentially topologically indistinguishable from (or topologically equivalent—i.e., homeomorphic—to) the original continuum X .

It must be also stressed here that in Sorkin (1991) a key attribute of the P_i s—one that enables one set up the projective system $\overleftarrow{\mathcal{P}}$ in the first place and then take its inverse limit—is that continuous surjections f_i , corresponding to ‘canonical’ projection maps from X to the \mathcal{U}_i -moduli spaces P_i (Sorkin, 1991), enjoy a *universal mapping property* that can be expressed by the diagram below:

$$\begin{array}{ccc}
 X & \xrightarrow{f_j} & P_j \\
 & \searrow f_i & \downarrow f_{ji} \\
 & & P_i
 \end{array} \tag{13}$$

That is, $f_i = f_{ji} \circ f_j$, and it reads that *the map (canonical projection) of X onto the finitary substitutes is universal among maps into T_0 -spaces*, with f_{ji} the *unique map*—itself an order-monotone surjection of P_i onto P_j ³⁰—mediating between the continuous projections f_i and f_j of X onto the T_0 -posets P_i and P_j , respectively. With these ‘canonical’ continuous projections of X onto the P_i s, the said inverse system of finitary posets can be written as a collection of triplets $\overleftarrow{\mathcal{P}} := \{(P_i, f_i, f_{ji})\}$; while formally, the inverse limit result above can now be cast as $P_\infty = \lim_{\leftarrow} f_{ji}(P_i) \overset{\text{homeo.}}{\simeq} X$ (modulo Hausdorff reflection). This universal mapping property of the maps between the finitary T_0 -posets is completely analogous to the one possessed by the differential triad morphisms (push-outs and pull-backs) mentioned earlier. In fact, in the paragraph after the next, when we will discuss finitary differential triads and their inverse limits, the ideas of Sorkin and Papatriantafyllou will appear to be tailor-cut for each other; albeit, with the

²⁹ The adverb ‘practically’ above pertains to the result from Sorkin (1991) that, at the inverse limit of $\overleftarrow{\mathcal{P}}$, one does *not* actually recover the topological manifold X itself, but a non-Hausdorff space P_∞ which includes X as a dense subset. However, one can get back X from P_∞ , by a procedure commonly known as *Hausdorff reflection*, as the set of the latter’s closed points (Kopperman and Wilson, 1997).

³⁰ Which, as noted earlier, corresponds to the act of topological coarse-graining $\mathcal{U}_i \leq \mathcal{U}_j$ ($i \leq j$ in some ‘refinement index-net’).

ADG-based work of Papatriantafillou adding an important *differential* geometric twist to Sorkin’s originally purely topological ideas.

3.2. Incidence Algebras of Finitary Posets: Differential Structure (‘Smoothness’) from Algebra

In a pair of papers in collaboration with Zapatrin (Raptis and Zapatrin, 2000, 2001), a so-called *incidence Rota algebra* Ω_i was associated, by *Gel’fand duality*, with every P_i . One formally writes the correspondence as:

$$P_i \longrightarrow \Omega_i(P_i) \tag{14}$$

The Ω_i s³¹ were seen to be unital, associative, but in general non-commutative,³² \mathbb{K} -algebras, which *a fortiori* are \mathbb{Z}_+ -graded discrete differential algebras (manifolds)

$$\Omega_i = \bigoplus_{n \in \mathbb{Z}_+} \Omega_i^n = \overbrace{\Omega_i^0}^{\mathcal{A}_i} \oplus \overbrace{\Omega_i^1 \oplus \Omega_i^2 \oplus \dots}^{\mathcal{R}_i} \equiv \mathcal{A}_i \oplus \mathcal{R}_i \tag{15}$$

with \mathcal{A}_i an abelian subalgebra of Ω_i ³³ and \mathcal{R}_i a graded differential \mathcal{A}_i -module.³⁴ Indeed, there is a discrete version d_i of the usual nilpotent Cartan-de Rham-Kähler differential operator effecting \mathbb{K} -linear grade-raising transitions of the sort $d_i : \Omega^n \longrightarrow \Omega^{n+1}$.

The careful reader will have perhaps noticed the following apparent discrepancy here: While Sorkin’s P_i s were purely discrete *topological* structures, their Gel’fand-dual picture in terms of the Ω_i appears to encode additional information about the *differential* structure (of the original continuum X that Sorkin started with). How did ‘differentiability’ (differential structure) creep into our considerations when, following Sorkin, the original investigations pertained only to ‘continuity’ (:topological structure)? The reason is that the P_i s can be also thought of as *homological* objects—as a matter of fact, as *simplicial decompositions* of the original manifold X . That is to say, the P_i s can alternatively (and equivalently) be viewed as *simplicial complexes* \mathcal{K}_i , and as a result, their corresponding incidence algebras as *incidence algebras of simplicial complexes* $\Omega_i(\mathcal{K}_i)$ (Raptis and

³¹ From now on we drop the (P_i) arguments from the Ω_i s.
³² They are abelian when the P_i s are discrete (i.e., completely disconnected, trivially Hausdorff) topological spaces.
³³ \mathcal{A}_i , generated by the ‘self-incidences’ (i.e., the reflexive relations of the points) in the underlying poset P_i , is a discrete analogue of the algebra $C^\infty(M)$ of coordinates (or of points, by Gel’fand duality/spectral theory) on a smooth manifold M .
³⁴ \mathcal{R}_i is a discrete analogue of the classical $C^\infty(M)$ -module of smooth differential forms on a differential manifold M . Each Ω_i^n in \mathcal{R}_i is a linear subspace of Ω_i .

Zapatrin, 2000, 2001; Zapatrin, 2002).³⁵ The d_i s of the Ω_i s can now be expressed in terms of the nilpotent homological boundary δ (border) and coboundary δ^* (coborder) operators (Zapatrin, 2002).³⁶ The dual character of the Ω_i s relative to the \mathcal{K}_i s can now be understood simply by noting that the former's elements are *cohomological* entities—i.e., discrete differential form-like objects, which are obviously dual to the homological simplices in the \mathcal{K}_i s.

3.3. Finitary Differential Triads

The observation above that the Ω_i s encode not only topological, but also *differential geometric* information (coming from X), motivated this author to try to apply the ADG-machinery to a finitary setting. But for that, some *sheaf-like* structure was needed to be introduced first.³⁷ Thus, *finitary spacetime sheaves* (finsheaves) \mathcal{S}_i over Sorkin's finitary posets were introduced and developed in Raptis (2000b). Originally, finsheaves were conceived, in complete analogy to the P_i s, as genuine finitary approximations of the sheaf \mathcal{C}_X^0 of continuous (\mathbb{R} -valued) functions on the topological manifold X , again in the sense that an inverse system thereof possessed a projective limit sheaf that is topologically indistinguishable from \mathcal{C}_X^0 . However, the original intention to build *differential*, not just topological, structure into the finsheaves mandated that this author should define *finsheaves of incidence algebras*. This definition was straightforward to arrive at since it was realized early on that the map (14) is, by construction,³⁸ a *local homeomorphism—alias*, a *sheaf* (Mallios, 1998a,b). Thus, finsheaves of incidence algebras Ω_i —essentially, the sheaf-theoretic localizations of the Ω_i s over Sorkin's P_i s—were introduced, and hence the ADG-theoretic panoply was ready to be used in the finitary realm.

³⁵ Indeed, the order n of each Ω^n in (15) corresponds to the *simplicial degree* (or cardinality) of the respective simplex in \mathcal{K}_i (Raptis and Zapatrin, 2000).

³⁶ In categorical terms, the simplicial analogue of the correspondence (14), $\mathcal{K} \longrightarrow \Omega_i$, turns out to be a (contravariant) functor between the category of (finitary) simplicial complexes and simplicial maps (or equivalently, the category of finitary posets and poset morphisms—i.e., order preserving/monotone maps), and the category of (finitary) incidence algebras and algebra homomorphisms (Raptis and Zapatrin, 2000, 2001; Zapatrin, 2002).

³⁷ The motivation mentioned above was a *mathematical* one. The *physical* motivation was that this author ultimately wished to localize or gauge (thus dynamically vary and 'curve') quantum causality (i.e., the incidence algebras modelling qausets) (Mallios and Raptis, 2001, 2003; Raptis, 2003). In turn, the act of 'localization' or 'gauging' is (mathematically) tautosemous to 'sheafification' (Mallios, 2004a), followed by endowing the resulting sheaf with a connection \mathcal{D} (Mallios, 1998a,b).

³⁸ The construction alluded to above was coined *Gel'fand spatialization* in Raptis and Zapatrin (2000, 2001) (see also Zapatrin (1998)), whereby roughly, the 'local' Sorkin order-topology of P_i is equivalent to the 'local' Rota topology assigned to the (primitive) spectra of the Ω_i s, a procedure which is effectively an application of Gel'fand duality to the finitary realm of the P_i s.

Indeed, finsheaves of incidence algebras define (graded) *finitary differential triads*

$$\mathfrak{T}_i := \left(\mathbf{A}_i \equiv \mathcal{A}_{P_i}, d_i, \mathbf{R}_i \equiv \mathcal{R}_{P_i} = \bigoplus_{n \geq 1} \Omega_i^n \right)^{39} \tag{16}$$

which have been seen to carry, virtually unaltered, to the ‘discrete,’ finitary setting certain key results of the CDG of smooth manifolds, such as the *Poincaré lemma*, the *de Rham theorem*, the *Weil integrality theorem*, the *Chern-Weil theorem*, and much more (pertaining, for example, to geometric prequantization of gravity in an ADG-setting) (Mallios and Raptis, 2002).

Those applications aside for a moment, at this point we would like to close this paragraph by giving a characteristic example of the aforementioned categorical versatility of ADG, as opposed to the rigidity of the manifold-based CDG and of the category *Man* underlying it. To this end, we show how one can arrive straightforwardly from Sorkin’s finitary posets to finitary differential triads without having to go the long laboriously ‘constructive’ way via simplicial complexes, their Gel’fand-dual incidence algebras and the finsheaves thereof.⁴⁰ This involves an immediate application to the Sorkin scheme of the push-out and pull-back (along continuous maps between base topological spaces) results mentioned in the previous section, as follows:

- First, unlike Sorkin whose considerations in Sorkin (1991) were purely topological as noted earlier, we assume that (the region of) the manifold X carries not only the usual topological (\mathcal{C}^0), but also the standard differential (\mathcal{C}^∞) structure of a locally Euclidean space. What amounts to the same from an ADG-theoretic vantage, we suppose that X supports the classical \mathbf{K} -algebraized space $\mathbf{D}_\infty := (X, \mathcal{C}_X^\infty)$ carrying the classical differential triad $\mathfrak{T}_\infty := (\mathcal{C}_X^\infty, \partial, \Omega_X)$ of a differential manifold.
- Then, we factor *à la* Sorkin X by $\overset{\mathcal{U}_i}{\sim}$ to obtain the finitary substitute P_i (11) and, as a result, the continuous surjection f_i between them (13).

³⁹ Again, like before, from now on we will omit the base space P_i subscript from the finsheaves involved, but we will retain the ‘finitarity’ or resolution index ‘ i ’ to be used in the projective and inductive limits subsequently. Also note that built into \mathfrak{T}_i are higher order (or grade) extensions Ω^n of the Ω^1 appearing in the abstract differential triad in (4), as well as higher order prolongations d_i^n ($n \geq 1$) of $\partial_i \equiv d_i^0$, which \mathbf{K} -linearly map Ω_i^n to Ω_i^{n+1} (Mallios and Raptis, 2001, 2002, 2003). The latter will be generically represented by the finitary version d_i of the Cartan-de Rham-Kähler (exterior) differential.

⁴⁰ The reader should note that in the past trilogy (Mallios and Raptis, 2001, 2002, 2003) of finitary applications of ADG, we indeed followed that ‘roundabout’ way in order to define finitary differential triads.

- Finally, we evoke the push-out result of Papatriantafillou and endow P_i with the differential triad $f_{i*}(\mathfrak{T}_\infty)$, which can be readily identified with the finitary differential triad \mathfrak{T}_i of (16).

This f_{i*} -induction of the (essentially algebraic—i.e., sheaf-theoretic) differential geometric mechanism from \mathfrak{T}_∞ on the continuum X to \mathfrak{T}_i on the ‘discretum’ P_i has been recently coined the ‘*Newtonian spark*’ in Mallios (2004b); Mallios and Raptis (2004) and it exemplifies what we regard as being *the* subtle epitome of ADG, namely, that although we may initially inherit the (essentially algebraic) differential geometric mechanism—in essence, the differential d —from a base space (here, be it a locally Euclidean one such as X), we then ‘forget’ about that background sheaf-theoretic ‘localization scaffolding’ and develop all the various differential geometric constructions ‘algebraically in the stalk’ (i.e., with the algebraic objects living in the relevant sheaves’ spaces—or what is the same, solely with the relevant sheaves’ sections), and what’s more, completely independently of that surrogate X , which just furnished us with the invaluable for actually doing *differential geometry* d .

3.4. Finitary Vacuum Einstein Equations

It has been shown (Mallios and Raptis, 2003) that with the \mathfrak{T}_i s and the general ADG-machinery in hand, one can transcribe to the finitary realm all the ideas and constructions of the manifold-based (pseudo)-Riemannian geometry that we recalled in Section 2. That is, one can develop a ‘*finitary Riemannian geometry*,’ which is a particular instance of the background manifoldless Riemannian geometry of Section 2. In particular, one can formulate on each \mathfrak{T}_i a finitary version of the vacuum Einstein Equations (9), reading:

$$\mathcal{R}_i(\mathcal{E}_i) = 0 \tag{17}$$

with \mathcal{R}_i the finitary version of the Ricci scalar, and \mathcal{E}_i the $*$ -dual of the finsheaf Ω_i of incidence algebras, as posited by ADG.

Having the \mathfrak{T}_i s and (17) holding on each of them at our disposal, in the next section, we take on their inverse and direct limits.

4. THE CATEGORY OF DIFFERENTIAL TRIADS IS BICOMPLETE

That \mathfrak{T} is bicomplete is in fact just a result (theorem) in the category-theoretic perspective on ADG (Papatriantafillou, 2000, 2001, 2004, 2003, 2005), but due to its importance in the present paper, we promote it to the title of the present section. Indeed, as noted earlier, \mathfrak{T} is closed under both projective and inductive limits. This means that inverse and direct systems of differential triads possess categorical limit and colimit spaces that are themselves differential triads. Since

$\mathfrak{D}\mathfrak{T}_i$ is a subcategory of $\mathfrak{D}\mathfrak{T}$, projective and inductive systems of \mathfrak{T}_i s have inverse and direct limit structures that are themselves triads; albeit, not necessarily *finitary*, ‘discretum’ ones. In fact, as we shall see in the next paragraph, the limit triads that we are interested in and are of significance for the physical interpretation of our theoretical scheme are ‘*infinitary*,’ ‘*classical continuum*’ ones.

4.1. Inverse and Direct Limits of \mathfrak{T}_i s and Their Vacuum Einstein Equations

We first start with Sorkin’s result noted earlier, namely, that the inverse system $\overleftarrow{\mathcal{P}}$ of the P_i s has, at the limit of maximum (topological) resolution or refinement of the \mathcal{U}_i s, a projective limit space P_∞ that for all intents and purposes is topologically indistinguishable from (i.e., homeomorphic to) the C^0 -continuum X that we started with. Likewise for the analogous finsheaf-discretizations of C^0_X in Raptis (2000b).

As it has already been pointed out numerous times in the past trilogy (Mallios and Raptis, 2001, 2002, 2003), since the Ω_i s are categorically dual to the P_i s, one infers that they too comprise, dually now, a *direct* system $\overrightarrow{\mathfrak{R}} = \{\Omega_i\}$ possessing an inductive limit incidence algebra which, in view of the fact that the Ω_i encode information not only about the topological, but also about the differential, structure of the continuum X , should come close to emulating the classical differential geometric structure of X —namely, the $C^\infty(X)$ -module of differential forms on the differential manifold X .⁴¹ Accordingly, passing to finitary (‘discretum’) differential triads, they also constitute a projective/inductive system $\overleftarrow{\mathcal{T}}$ ⁴² possessing, according to Papatriantafillou’s results above, at the infinite limit of resolution (refinement) of the \mathcal{U}_i s, an ‘infinitary’ (continuum) triad \mathfrak{T}_∞ which comes as close as possible (via Sorkin’s scheme) to the classical one $\mathfrak{T}_\infty = (C^\infty_X, \partial, \mathbf{O}mg_X)$ supported by the differential manifold X .

The expression ‘comes as close as possible to \mathfrak{T}_∞ ’ above pertains to the fact that, much in the same way that one does not actually recover X as the inverse limit space of $\overleftarrow{\mathcal{P}}$, one also does not exactly get C^∞_X and the $C^\infty(X)$ -module sheaf \mathfrak{Q} of (germs of) smooth differential forms (over the differential manifold X ’s points) at the direct limit of (infinite localization of) the Ω_i s in $\overrightarrow{\mathfrak{R}}$. Rather, similarly to the fact that one gets a ‘larger’ inverse limit topological space P_∞ having X as a dense subset in Sorkin’s scheme (i.e., roughly, P_∞ has more points than the original X), one anticipates $\overrightarrow{\mathfrak{R}}$ to yield at the inductive

⁴¹ This has been investigated in detail in Raptis and Zapatrin (2000, 2001); Zapatrin (2002, 2001).

⁴² The joint epithet ‘projective/inductive’ to $\overleftarrow{\mathcal{T}}$ pertains exactly to the duality mentioned above: while the P_i s—the base spaces of the \mathfrak{T}_i s—constitute an inverse system $\overleftarrow{\mathcal{P}}$, their categorically dual Ω_i s—inhabiting the stalks of the finsheaf spaces in the \mathfrak{T}_i s—constitute a direct system $\overrightarrow{\mathfrak{R}}$. Informally-syntactically speaking, \mathcal{U}_i -refinement for the P_i s goes from-right-to-left, while for the Ω_i s from-left-to-right. (See expression (150) in Mallios and Raptis (2003).)

limit a ('topological') algebra \mathcal{A}_∞ 'larger' than $C^\infty(X)$ and consequently an \mathcal{A}_∞ -module \mathcal{R}_∞ of differential form-like entities 'larger' than the standard C^∞ -one. In Zapatrin's words, when he was working out continuum limits of incidence algebras of simplicial complexes (Zapatrin, 2002, 2001): "*it is as if too many functions and forms want to be smooth in the continuum limit.*"⁴³ One intuits that much in the same way that Hausdorff reflection gets rid of the 'extra points' of P_∞ to recover the C^0 -manifold X , so by ridding Ω_∞ of the 'rogue' extra functions and forms on P_∞ (e.g., by factoring it by a suitable differential ideal; Zapatrin (2002, 2001)), one should recover the usual smooth functions and forms over the differential manifold X . Nevertheless, the important point for the exposition here is that *one does indeed get a continuum differential triad*, which however, only in order to be formally distinguished from the classical C^∞ -smooth one \mathfrak{T}_∞ to avoid any minor technical misunderstanding, one might wish to call ' \mathcal{C} -smooth' and symbolize it by T_∞ (Mallios and Raptis, 2003). On the other hand, after having alerted the reader to this slight distinction between T_∞ and \mathfrak{T}_∞ , in the sequel, for all practical purposes and in order to avoid proliferation of redundant symbols, we shall abuse language and assume that T_∞ and \mathfrak{T}_∞ are 'essentially isomorphic' (i.e., effectively equivalent and indistinguishable). So, both will be generically referred to as *the classical continuum differential triad* (CCDT), with the symbols \mathfrak{T}_∞ and T_∞ used interchangeably.

Thus, we formally write for this joint inverse/direct limit procedure exercised on $\overline{\mathcal{T}}$

$$\lim_{\substack{i \rightarrow \infty \\ \infty \leftarrow i}} \overline{\mathcal{T}} = T_\infty \equiv \mathfrak{T}_\infty \tag{18}$$

As argued and shown in detail in Mallios and Raptis (2003), each fcq -differential triad \mathfrak{T}_i carries on its shoulders the whole ADG-machinery and structural panoply involved in the usual manifold-based (pseudo)-Riemannian geometry. In particular, (17) holds on each \mathfrak{T}_i and hence this information carries to the inverse/direct limit of $\overline{\mathcal{T}}$, which in turn is seen to support an inverse system $\overleftarrow{\mathcal{E}}$ of fcq -vacuum Einstein equations.⁴⁴ We thus recover a smooth continuum limit version of the vacuum Einstein equations, holding on T_∞ (or equivalently, on \mathfrak{T}_∞), which we formally depict as:

$$\lim_{\infty \leftarrow i} \overleftarrow{\mathcal{E}} = \lim_{\infty \leftarrow i} \mathcal{R}_i(\mathcal{E}_i) = \mathcal{R}_\infty(\mathcal{E}_\infty) = 0 \tag{19}$$

with \mathcal{E}_∞ the dual of the C_X^∞ -module sheaf of (germs of) smooth differential forms on the differential manifold X comprising the CCDT \mathfrak{T}_∞ , and \mathcal{R}_∞ the classical smooth (i.e., $\mathbf{A} \equiv C_X^\infty$ -valued) Ricci curvature scalar.

⁴³ Roman Zapatrin in private e-mail correspondence.

⁴⁴ For example, again see expression (150) in Mallios and Raptis (2003).

4.2. ‘Correspondence Limit/Principle’ Interpretation of Inverse/Direct Limits

We briefly remark here that in Raptis and Zapatrin (2000, 2001), in view of the quantum interpretation that the Ω_i s enjoy, the continuum inverse limit of $\overleftarrow{\mathcal{P}}$, and dually, the direct limit of $\overrightarrow{\mathfrak{A}}$, was interpreted as *Bohr’s correspondence principle*, otherwise known as *the classical continuum limit*. As a result, (19) may be interpreted as follows: *at the continuum limit of infinite topological resolution (or refinement) of X into its points, or equivalently, of infinite sheaf-theoretic localization of the incidence algebras over X ’s points, one obtains the classical continuum vacuum Einstein equations⁴⁵ from the individual fcq -ones holding on each ‘discretum’ triad \mathfrak{T}_i (17)*. In other words, and this will prove to be of importance for some remarks that we are going to make in the next two sections regarding the application of ADG to both classical and quantum gravity, *ADG, and the vacuum Einstein gravity to which it has been applied so far, is genuinely background spacetime-independent, i.e., the vacuum Einstein equations are in force independently of whether one assumes the base space(time) to be a ‘quantal discretum’⁴⁶ or a ‘classical continuum.’⁴⁷*

5. FINITARY-ALGEBRAIC EVASION OF THE INTERIOR SCHWARZSCHILD SINGULARITY

We have now built a sufficient conceptual and technical background to present in a straightforward fashion the ADG-theoretic evasion of the inner Schwarzschild singularity entirely by finitistic and algebraic means. First however, in order to present that ‘resolution’ in a more effective way, we recall a contrasting theory of the interior Schwarzschild singularity. This is the standard one based on the usual approach to GR via CDG, the C^∞ -smooth base spacetime manifold and the smooth Lorentzian metric on it (i.e., *in toto*, the classical pseudo-Riemannian geometry underlying GR). The following are well known, amply worked out facts about the Schwarzschild solution of the Einstein equations, which we thus present rather briefly and informally.

⁴⁵That is, the vacuum Einstein equations holding over the entire smooth manifold X —i.e., on \mathfrak{T}_∞ .

⁴⁶As it were, when (locally at least) only a finite number of ‘degrees of freedom’ of the vacuum gravitational field are excited (i.e., when only a locally finite number of events are involved, or dually/functionally, when only a finite number of ‘modes’ of the gravitational field ‘contribute’ to/‘participate’ at the gravitational dynamics at each spacetime event), and when some sort of quantization has already taken place (Raptis and Zapatrin, 2000, 2001).

⁴⁷That is, when the gravitational field ‘triggers’ or ‘excites’ a continuous infinity of spacetime events in the manifold, and all ‘quantum interference’ (coherent quantum superpositions between the elements of the Ω_i s) has been lifted (Raptis and Zapatrin, 2000, 2001).

5.1. Classical Schwarzschild Preliminaries: The Standard View, the Usual Suspects and the Main Problematics

We begin by noting some familiar features of GR. First of all, the original theory was formulated in terms of the smooth metric tensor $g_{\mu\nu}$ on a differential spacetime manifold X . That is to say, the sole dynamical variable in GR (as originally formulated by Einstein) is $g_{\mu\nu}$, whose 10 independent $C^\infty(X)$ -valued components represent the *gravitational potentials* and at the same time they enter into the pseudo-Riemannian line element representing the *chrono-geometric structure* of the spacetime continuum. In a nutshell, $g_{\mu\nu}$ represents gravity-cum-background spacetime chrono-geometry, and the dynamical equations that it obeys in the absence of matter are the (vacuum) Einstein equations (19)—formally, non-linear (hyperbolic) second-order partial differential equations (PDEs) for $g_{\mu\nu}$.

The Schwarzschild solution of the said equations represents the spherically symmetric vacuum gravitational field outside a massive, spherically symmetric body of mass m . On grounds of physical import alone, our choice of this particular solution on which to exercise our ADG-machinery and results in order to ‘resolve’ it may be justified on the fact that experimentally all the differences between non-relativistic (Newtonian) gravity and GR have been based on predictions by this solution (Hawking and Ellis, 1973). Also, since comparison with Newtonian gravity allows us to interpret the Schwarzschild solution as the gravitational field (in empty spacetime) produced by a point-mass source m viewed from far away (i.e., from infinity) (Hawking and Ellis, 1973), Finkelstein’s original treatment of the Schwarzschild gravitational field as being produced by a point-particle in an otherwise empty spacetime manifold (Finkelstein, 1958) appears to be a good starting choice.

So first, following Finkelstein, one assumes that spacetime is a smooth (C^∞) or even an analytic (C^ω) manifold X ,⁴⁸ and then one places at its ‘center’ (:interior) a point-mass m . For a Cartesian coordinate system with m at its origin, the ‘effective’ spacetime manifold of this point-particle becomes X minus the particle’s ‘wristwatch’ time-line $L_t := \{p \in X : x_i(p) = 0, (i = 1, 2, 3, t \equiv x_0)\}$; that is to say,

$$X_S = X - L_t \quad (20)$$

with the subscript ‘S’ standing for ‘(S)chwarzschild.’ Then, one observes that m is the source of a gravitational field, represented by a smooth (or analytic)

⁴⁸ In this paper, we shall *not* distinguish between a C^∞ - and a C^ω -manifold (or for the same reason, between CDG and Calculus or Analysis). From an ADG-theoretic viewpoint, as noted earlier, a smooth manifold X corresponds to choosing C_X^∞ for structure sheaf, while an analytic one has $\mathbf{A} \equiv C_X^\omega$ —the structure sheaf of coordinate functions (of X ’s points) each admitting analysis (expansion) into power series. Admittedly, C^ω - is a slightly stronger assumption for a manifold than C^∞ -, but this does not change or inhibit the points we wish to make here about the Schwarzschild singularity and its bypass in the light of ADG.

spacetime metric $g_{\mu\nu}$, which obeys the vacuum Einstein equations (19). The Schwarzschild solution of the equations (19) is the Schwarzschild metric $g_{\mu\nu}^S$ expressed in Cartesian-Schwarzschild coordinates, which in turn defines an infinitesimal proper time increment, as follows:

$$ds_S^2 = (1 - r_S^{-1})(dx_S^0)^2 - (1 - r_S^{-1})^{-1}dr_S^2 - (dx_S^i dx_S^i - dr_S^2) \quad (21)$$

expressed in normalized, ‘natural’ units in which the so-called Schwarzschild radius ($r = 2m$) and the speed of light ($c = 10^8 \text{m/s}$) are equal to 1.⁴⁹

Evidently, $g_{\mu\nu}^S$ has two singularities: One right at the *locus* of the point-mass—the Cartesian origin ($r = 0$), and one at the Schwarzschild radius ($r = 1$) delimiting a space-like 3-dimensional unit-spherical shell in X , commonly known as the Schwarzschild horizon. The two singularities are usually pitched as the *interior* (inner) and *exterior* (outer) Schwarzschild singularities, respectively.⁵⁰

Finkelstein (1958) initially considered an analytic metric $g_{\mu\nu}^F$ on X , expressed in what is nowadays usually called Eddington–Finkelstein coordinates,⁵¹ defining the following infinitesimal spacetime line element

$$\begin{aligned} ds_F^2 &= (1 - r_F^{-1})(dx_F^0)^2 + 2r_F^{-1}dx_F^0 dr_F - (1 + r_F^{-1})dr_F^2 - (dx_F^i dx_F^i - dr_F^2) \\ &= -\left(1 - \frac{2m}{r}\right)(dn^\pm)^2 \pm 2dn^\pm dr + r^2(d\theta^2 + \sin^2\theta d\phi^2) \end{aligned} \quad (22)$$

and he then showed that, for the region of X outside the Schwarzschild horizon 3-shell ($r_F > 1$), the following simple ‘logarithmic time coordinate change’ from the analytic Finkelstein ${}^\omega\mathbf{A}_F = {}^\omega(x_F^\mu)$ coordinates to the also analytic Schwarzschild ones ${}^\omega\mathbf{A}_S = {}^\omega(x_S^\mu)$

$$\begin{aligned} &{}^\omega\mathbf{A}_F \longrightarrow {}^\omega\mathbf{A}_S : \\ x_F^0 &\longrightarrow x_S^0 = x_F^0 + \ln(r_F - 1) \\ x_F^i &\longrightarrow x_S^i = x_F^i \end{aligned} \quad (23)$$

⁴⁹ Also, in (21) above, $r_S = \sqrt{x_S^i x_S^i}$ and $dr_S = r_S^{-1}x_S^i dx_S^i$. The more familiar (i.e., not in natural units) expression for the Schwarzschild line element in cartesian coordinates is $(1 - \frac{2m}{r})dt^2 + dx^2 + dy^2 + dz^2 + \frac{2m}{r(r-2m)}(xdx + ydy + zdz)^2$, while in spherical-Schwarzschild coordinates (again not in natural units), it reads $-(1 - \frac{2m}{r})dt^2 + (1 - \frac{2m}{r})^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$.

⁵⁰ The Schwarzschild horizon is the horizon of the Schwarzschild black hole, and it is supposed to host the inner Schwarzschild singularity at its kernel, ‘beyond the horizon.’

⁵¹ The Eddington–Finkelstein frame consists of so-called *logarithmic-null spherical coordinates* (n^\pm, r, θ, ϕ), with the null coordinate n^\pm being either *advanced* $n^+ := t + r'$ or *retarded* $n^- := t - r'$, and r' defining a logarithmic radial coordinate $r' := \int \frac{dr}{1-2mr^{-1}} = r + 2m \log(r - 2m)$.

transforms the line element ds_F^2 (and the associated $g_{\mu\nu}^F$) in (20) to the Schwarzschild one ds_S^2 (and its associated $g_{\mu\nu}^S$) in (21).

Conversely, he argued that since \mathcal{R}_∞ in (19) is a tensor with respect to the ${}^\omega\mathbf{A}_F$ coordinates, the vacuum Einstein equations hold in all X (now coordinatized by ${}^\omega\mathbf{A}_F$ ⁵²)—in particular, they hold on the Schwarzschild horizon unit-shell.

In toto, Finkelstein showed that the analytic coordinate change

$$X_S \equiv (X, {}^\omega\mathbf{A}_S) \longrightarrow (X, {}^\omega\mathbf{A}_F) \equiv X_F \quad (24)$$

amounts to an *analytic extension* of X_S (coordinatized by the Cartesian ${}^\omega\mathbf{A}_S$ and carrying the analytic $g_{\mu\nu}^S$ defining ds_S^2 above—which is singular at $r = 1$), to X_F (coordinatized by the analytic ${}^\omega\mathbf{A}_F$ and carrying the analytic $g_{\mu\nu}^F$ defining ds_F^2 , which is not singular at the Schwarzschild radius!).

In fact, Finkelstein showed that the said analytic extension of X_S to X_F can be carried out in two distinct ways,⁵³ each one being the time-reversed picture of the other, which in turn means that the $r = 1$ Schwarzschild horizon, far from being a real singularity, acts as “*a true unidirectional membrane*” in the sense that “*causal influences can pass through it only in one direction*” and, moreover, he gave a particle-antiparticle interpretation of this gravitational time-asymmetry (Finkelstein, 1958).⁵⁴

On the other hand, about the inner Schwarzschild singularity Finkelstein concluded that the theory (i.e., the manifold and CDG-based GR) is out of its depth as there is no (analytic) coordinate change that can remove it like the outer one. In other words, the interior Schwarzschild singularity, right at the point-particle m , is regarded as being a ‘genuine,’ ‘true’ singularity of the gravitational field, not removable (or resolvable) by analytic (i.e., CDG-theoretic) means (Finkelstein, 1958; Hawking and Ellis, 1973; Clarke, 1993).⁵⁵ Which brings us to the general consensus about ‘true,’ as opposed to merely ‘virtual’ or ‘coordinate,’ \mathcal{C}^∞ -spacetime singularities.

⁵² Which we may just as well symbolize by X_F .

⁵³ Depending on whether one chooses advanced or retarded logarithmic-null coordinates.

⁵⁴ The null (in the Finkelstein frame) hypersurface Schwarzschild horizon is also known as a *closed trapped surface* (Hawking and Ellis, 1973), which ‘traps’ past- (resp. future-) directed causal (i.e., time-like or null) signals depending on whether one chooses advanced (resp. retarded) Finkelstein coordinates to chart the original manifold. Also, it can be easily seen that inside Schwarzschild horizon the original time and radial coordinates exchange roles.

⁵⁵ Indeed, in the n^+ -picture, any future-directed causal curve crossing the Schwarzschild horizon can reach $r = 0$ in finite affine parameter distance (see next paragraph). Moreover, it can be shown that as $r \rightarrow 0$ the Ricci scalar curvature \mathcal{R} in (19) blows up as $\frac{m^2}{r^6}$, while there is no further analytic extension (even in a C^2 -differential, or even in a C^0 -topological, fashion!) of the Schwarzschild spacetime manifold across the $r = 0$ locus.

5.2. ‘True’ Versus ‘Coordinate’ Singularities: A CDG-Conservatism and Monopoly Underlying All Approaches (So Far) to Spacetime Singularities

The two Schwarzschild singularities above provide a good example of the general way we think about and actually deal with gravitational singularities in the CDG and manifold-based GR.

To begin with, it must be stressed up-front that *there is no general, concise and ‘rigorous’ definition of (‘true’) singularities in GR* (Geroch, 1968; Hawking and Ellis, 1973; Clarke, 1993; Rendall, 2005). Rather, one proceeds by *elimination and exclusion* in order to identify genuine gravitational singularities and separate them from ‘apparent,’ coordinate ones, in the following way. Given a singular gravitational spacetime—by which one means a manifold M (of a certain order of differentiability⁵⁶) endowed with a (Lorentzian) metric g (of maximum order of differentiability assumed for the underlying M) satisfying Einstein’s equations and possessing singularities at certain *loci* of M —one tries to analytically (or anyway, smoothly, or in a C^k -fashion) extend M ⁵⁷ past those *loci* so as to include them with the other ‘regular’ points of M . If there happens to be such an extension, the singularity in focus is regarded as an ‘apparent,’ ‘virtual,’ *coordinate* one—an indication that the physicist originally chose an inappropriate system of coordinates (patches) to chart M and to express $g_{\mu\nu}$ with respect to it. The exterior Schwarzschild singularity we saw earlier is the archetypical example of such a coordinate singularity. On the other hand, if there is no such extension, the ‘anomalous’ *locus* is branded a ‘true,’ ‘real,’ ‘genuine’ singularity. The inner Schwarzschild singularity is the archetypical example of such a real singularity, in the vicinity of which g_S (and the Ricci scalar) diverges to infinity. Kruskal’s maximal analytic extension of X_S above did not manage to include it with the other regular points of the spacetime manifold (Kruskal, 1960). Coordinate singularities are *not* considered to be ‘physical singularities’ (i.e., they are not of physical significance), while genuine ones *are* (Geroch, 1968; Hawking and Ellis, 1973; Clarke, 1993; Rendall, 2005).

Clearly then, coordinate singularities are regarded as being ‘regular points in disguise,’ and the differential manifold, together with the differential equations of Einstein that it supports, are still in force since they can be continued past them. On the other hand, this is not so for true singularities. The latter are *loci* where the differential law of gravity appears to stop (i.e., it ceases to hold) somehow, or even more graphically, it *breaks down*. Genuine singularities are sites where CDG (and the smooth manifold supporting its constructions) has reached the limit of

⁵⁶That is, an analytic ($\mathbf{A} \equiv C_M^\omega$), or smooth (C_M^∞), or even a manifold of finite order of differentiability (C_M^k).

⁵⁷Here, ‘to extend M ’ means essentially ‘to change coordinate structure sheaf of differentiable functions on M .’

its applicability and validity. Thus, let us recall briefly from Hawking and Ellis (1973); Clarke (1993) the three general kinds of gravitational singularities, and what underlies them all. We shall first mention *en passant* how one usually copes with genuine singularities in a manifestly CDG-conservative fashion.

Apart from *analytic inextendibility*, the other ‘defining’ feature of real spacetime singularities is (*causal geodesic incompleteness*). Roughly, the idea behind *spacetime incompleteness* is that (material) particles cannot reach (genuine) singularities in ‘finite (proper) time’ by following, under the focusing action of the strong gravitational field at the purported singular *loci*, smooth (causal) paths (geodesics) in the manifold M . Historically, the importance of (causal—i.e., time-like and null) geodesic incompleteness was first recognized in Geroch (1968). Subsequently, null geodesic incompleteness was the central prediction of the celebrated singularity theorems of Hawking and Penrose (Hawking and Penrose, 1970; Hawking and Ellis, 1973). However, it is not entirely clear what spacelike incompleteness means physically, since spacelike curves in M do not have an interpretation as histories of physical objects (i.e., fields and their particles). On the other hand, as Clarke points out in Clarke (1993), one need not consider only ‘free falling’ observers following causal *geodesics*, since other physically admissible frames—ones with bounded acceleration for instance—may be able to reach the point-*loci* in question in finite (proper) time, even though geodesic observers cannot. In order to include the world-lines of such in principle arbitrarily accelerated observers, paths more general than geodesics—ones parameterized not by proper time, but by an arbitrary so-called *general affine parameter*—must also be included in the definition of incompleteness. *In toto*, incompleteness pertains to the idea that curves of finite (general affine parameter) length cannot reach the singular *loci* in question. The bottom-line of all this is that a *spacetime is called singular if it is incomplete and inextendible*, in the above sense.

Now there appears to be a clear-cut way to proceed in dealing with true spacetime singularities, namely, one can relegate them to the ‘edge’ of a maximally extended spacetime manifold and view them as ‘asymptotically terminal points’ of incomplete curves. That is, one thinks of genuine singularities as *loci* situated on a certain boundary set ∂M adjoined to M , with the latter endowed with an ‘appropriate’ topology, which in turn qualifies $\overline{M} = M \cup \partial M$ as *the closure of M* and recognizes ∂M as a topological boundary proper. Parenthetically, without going into any detail, so far there are two basic singular boundary constructions: The *causal boundary* of Geroch, Kronheimer and Penrose (Geroch *et al.*, 1972), and the so-called *b-boundary* of Schmidt (Schmidt, 1971). Each of these two boundaries (and associated topologies) has its own pros and cons that we do not want to go into here, but for a detailed exposition of and comparison between them the reader is referred to Hawking and Ellis (1973); Clarke (1993).

Having ascribed a topology and a boundary to the spacetime continuum, and concomitantly having pushed the genuinely singular *loci* out of the regular M and

virtually to ‘the margin of spacetime’ (i.e., onto ∂M), one identifies three general types of genuine C^∞ -gravitational singularities (Clarke, 1986):

1. **(Differential) geometric singularities (DGS):** boundary points for which there is no C^k -differential extension of (the metric on) M so as to remove them.
2. **(Various) energy singularities (VES):** boundary points for which there is no (analytic) extension of M that removes them satisfying at the same time various energy conditions (inequalities) (Clarke, 1993), the most prominent and generic ones being *gravitational energy positivity* (;gravity is always attractive) and the associated *weak and dominant energy conditions* (Hawking and Ellis, 1973).
3. **(Solution) field singularities (SFS):** boundary points for which there is no (analytic) extension of M that removes them and is a solution of the Einstein field equations in question (e.g., Einstein-scalar or the Einstein-fluid equations). The important thing to mention here is that the term *solution* to the field equations means *generalized smooth* or *smeared*—what is commonly known as *distributional*, solution.

Plainly, (*analytic*) *inextensibility*—loosely speaking, our inability to apply CDG or Analysis—*underlies all three ‘definitions’ of genuine singularities above*. Metaphorically speaking, *true singularities are breakdown points of the Differential Calculus*. We will come back to this point in the sequel.

In the present paper, we will be predominantly interested in DGSs—which incidentally are singularities of the ‘purest’ kind *visà-vis* differential geometric considerations (Clarke, 1986)—as they manifestly depict the aforesaid Calculus or ‘classical differentiability’ breakdown, as Clarke (1986) explicitly points out:

... Thus the definition of a [differential geometric] singularity depends on the definition of an [analytic] extension of [the] space-time [manifold], and so the question of what counts as a singularity depends on what sort of extension is allowed. We call a boundary point [of a smooth manifold] a class C^k [differential] geometrical singularity if there is no [analytic] extension with a C^k metric that removes it; i.e., if it is associated with a breakdown of differentiability of the metric at the C^k level ...”⁵⁸

It must be also noted here that the way in which we ascribe a topology and construct a boundary to M on which true singularities are located, apart from its physical motivation,⁵⁹ exemplifies in our opinion the general *CDG-conservative*

⁵⁸ In square brackets are our own additions for clarity and completeness.

⁵⁹ For example, understandably the physicist would like to have a ‘controlled’ study of the asymptotic behavior of, say, the Riemann curvature tensor (whose components represent gravitational tidal forces) as one approaches (i.e., in the immediate neighborhood of) the singularity in focus—e.g., one would like to have an analytic picture of the way the curvature diverges in the neighborhood of the singularity. By acquiring such an analytic picture, even if one does hope to ultimately remedy singularities, one at least wishes to understand better what is going on near a singularity (e.g.,

attitude regarding the appearance and treatment of singularities in GR, which we briefly explain now.

Judging by the way we try to define genuine singularities by elimination and technically (:mathematically) deal with them, on the one hand, physical spacetime events are identified with the regular points of M —those at which the differential equations of Einstein hold and they do not suffer from any ‘differential geometric disease’ (e.g., the differentiability of the solution metric does not break down in any sense at them as in the case of the DGSs above). On the other hand, genuine, physically interesting and significant singularities are pushed—as it were, by mathematical fiat—to the boundary of the spacetime continuum in order to preserve the CDG-machinery within the otherwise regular M . This is precisely what we refer to as the *manifold and, in extenso, CDG-conservatism* in our analysis of spacetime singularities (Clarke, 1993). Indeed, a self-referential pun is intended here: *the analysis of spacetime singularities is essentially the (manifold-based) Analysis applied to the study of (true) spacetime singularities, which, ‘by definition’—i.e., by the analytic inextensibility of M past them—ultimately resist Analysis (analytic extension).*

Genuine singularities, as opposed to merely coordinate ones, are *loci* where the manifold-based Calculus (Analysis) comes to an end and hence the manifold-based GR is out of its depth (i.e., the differential equations of Einstein appear to break down and lose their predictive power—e.g., the solution metrics blow up and ‘yield’ physically meaningless infinities for ‘observable’ quantities like the curvature tensor). There is a tension here: *physical* spacetime events, including coordinate singularities, are the regular points in the interior M of \overline{M} where CDG applies galore, but *physical* singularities are *loci* on ∂M where CDG fails to apply (breaks down), while, in a paradoxical sense, we seem to persistently employ CDG (Analytical) means to study the latter (Clarke, 1993). This CDG-conservatism may be simply understood and justified on the ground that *the only way we so far know how to do differential geometry is via (the ‘mediation’ in our calculations—in fact, in our Differential Calculus!—of) a background continuum space(time), a base differential manifold.*

However, in view of the CDG-monopoly above and with ADG in mind, we would like to draw a fine line here: while we agree that CDG (as a *mathematical* framework for doing differential geometry) becomes inadequate at true singularities, we cannot accept that the *physical* law of gravity (modelled after a *differential* equation) breaks down at a singularity all because we traditionally tend to identify *physical* spacetime with *our* mathematical model M which in turn vitally supports CDG. In this line of thought, Einstein’s words from Einstein (1956) immediately spring to mind:

classify singularities according to their strength (Clarke, 1993)) and perhaps achieve a better control of those unphysical divergences. To that end, so-called asymptotic growth boundary conditions at the vicinity of the singularity are usually prescribed (Clarke, 1993).

... A field theory is not yet completely determined by the system of field equations. Should one admit the appearance of singularities?... It is my opinion that singularities must be excluded. *It does not seem reasonable to me to introduce into a continuum theory points (or lines etc.) for which the field equations do not hold*⁶⁰

Of course, the distinction we drew above would be simply unfounded had we not have in our hands not only an alternative (to CDG) theoretical framework for doing differential geometry independently of a background M , but also had we not been able within this new framework to formulate Einstein’s equations as *differential* equations proper which, *a fortiori*, could be explicitly shown not to be impeded at all by (let alone break down in) the presence of singularities. ADG is that theoretical framework.

In connection with the above, a key observation in ADG is that, since as noted before a differential manifold is nothing else but the algebra $C^\infty(M)$ (or the structure sheaf $\mathbf{A} \equiv C_M^\infty$) of smooth coordinate functions on it (Gel’fand duality), and since all the singularities in the CDG and manifold-based GR are singularities of some smooth function on M , GR “*carries the seeds of its own destruction*” (Bergmann, 1979) in the form of singularities exactly because the physical laws that define it⁶¹ are mathematically represented by differential equations within the confines of the CDG-framework. What amounts to the same, the apparent ‘self-destructive’ feature of GR corresponding to smooth spacetime singularities is exactly due to the fact that *we have a priori* posited that *physical spacetime* is a differential manifold.

The crux of the argument here is that it is not the gravitational field and the law that it obeys that halt or even break down at a singularity as if they carry the seeds of their own inapplicability and downfall, but that it is precisely *our* mathematical means of effectuating (representing) that gravitational dynamics differential geometrically via the M -based CDG—i.e., via C_M^∞ carrying the germs of all smooth singularities—that mislead us into thinking that the CDG-based GR predicts its own autocatastrophe.⁶² And all this again because we persistently identify *physical* spacetime with a background locally Euclidean space. In other words, it is the *mathematical* notion of a ‘*base manifold*’ (CDG)—or equivalently, *our* choice of C_M^∞ for structure sheaf of coordinates—in the expression ‘*manifold-based GR*’ that carries the differential geometric anomalies in the guise of singularities that assail GR, and *not* the *physical* concept of gravitational field or even the differential equation that it obeys. Alas, the aforesaid CDG-monopoly and associated

⁶⁰ Our emphasis. . . .

⁶¹ We tacitly assume that *a physical theory is defined by the physical laws (:dynamical equations) formulated within the mathematical framework adopted by (and adapted to!) that theory*. As noted before, in the case of GR as originally formulated by Einstein, that mathematical framework was the CDG and manifold-based (pseudo)-Riemannian geometry.

⁶² And let it be stressed here that, in a ‘Popperian falsifiability’ sense, this is more often than not regarded as a virtue of GR.

conservatism has misled us into branding M as ‘*physical spacetime*’ and concomitantly made us (con)fuse *our* mathematical framework (CDG) with the *physical* theory itself (GR, gravity) to the extent that we coin genuine singularities as being *physical* ones.

Furthermore, here one could go as far as to maintain that ‘*Nature has singularities.*’ Precisely this we find hard to swallow: genuine singularities simply pronounce that the differential manifold-based CDG has ceased to be a good mathematical means (:language) for describing the physical law of gravity, and that therefore, an alternative mathematical framework (for doing *differential* geometry—provided of course that the physicist still wishes to represent physical laws by *differential* equations proper) must be sought after. What is implicit here is our general working philosophy that *whenever there appears to be a discord or asymphony between the mathematics and the physics, one should invariably question and try to modify the former, not the latter.* One should blame it on *our* maths, not on *Physics* (Mallios and Raptis, 2003, 2004).

5.3. The ADG-Theoretic Finitary-Algebraic ‘Resolution’ of the Inner Schwarzschild Singularity: A ‘Static,’ Spatial, Localized Point-Resolution

This is the neuralgic part of the present paper in which everything that we have been saying earlier synergistically comes to effect. We hereby present the finitistic-algebraic evasion of the interior Schwarzschild singularity—regarded as a ‘static,’ spatial, localized point-singularity—by ADG-theoretic means in the form of an outline of the steps of a ‘syllogism’ leading directly to that ‘resolution,’ as follows:

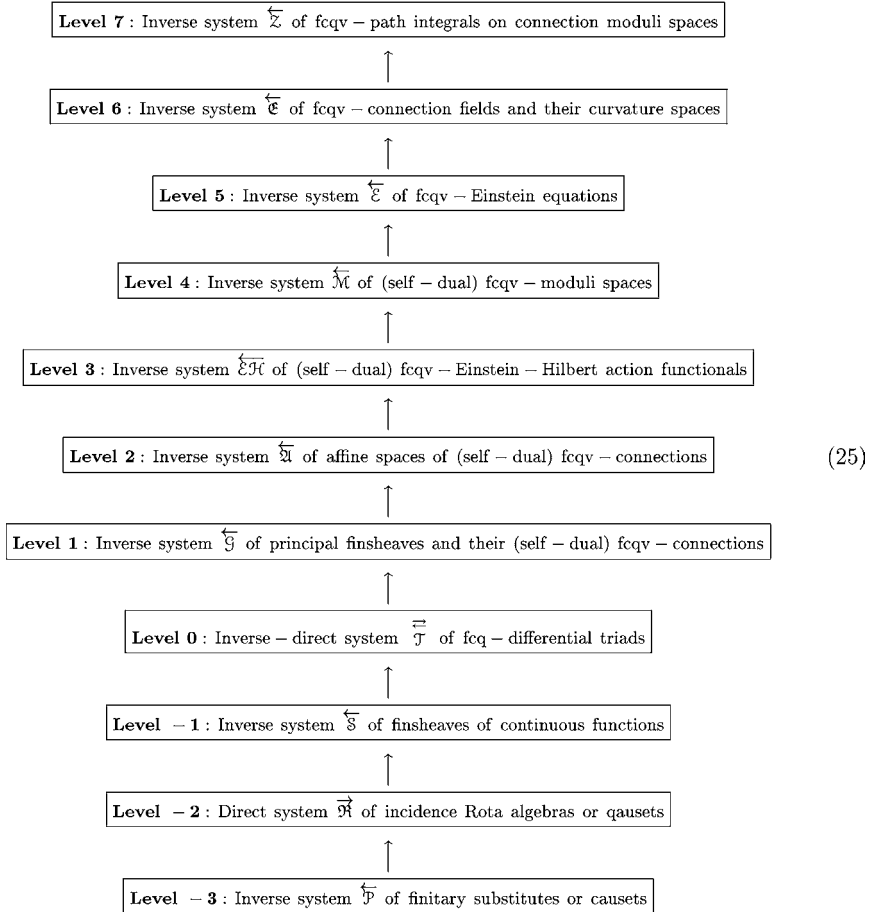
- First we consider an open and bounded region X of a spacetime manifold M , from which initially, *à la* Sorkin (Sorkin, 1991), we retain only its topological (i.e., C^0 -continuous) structure—that is, without *a priori* alluding to its differential (i.e., C^∞ -smooth) structure.
- We then let a point-particle of mass m be situated at the ‘center’ of X , as in Finkelstein (1958). That is, we assume that m is a point in X ’s interior without evoking any boundary ∂X construction.
- Next, we cover X by a locally finite open covering \mathcal{U}_i . In the jargon of ADG, the U s in \mathcal{U}_i are called ‘*open local gauges*’ (Mallios, 1998a,b, 2001; Mallios and Raptis, 2001, 2002, 2003).
- Subsequently, we first discretize X relative to \mathcal{U}_i in the manner of Sorkin (11), and then pass to the Gel’fand-dual representation of the resulting finitary posets P_i in terms of discrete differential incidence algebras Ω_i (14).

- Then we consider finsheaves (Raptis, 2000b) of incidence algebras Ω_i in the manner first introduced in Mallios and Raptis (2001). Parenthetically, one may wish to bring forth from Raptis (2000a) the causet and quaset interpretation that the P_i s and their associated Ω_i s may be given, as well as the finsheaves thereof (Mallios and Raptis, 2001).
- We then recall from (16) the finitary differential triads \mathfrak{T}_i (of quaset) that the said finsheaves define. Here the reader may like to remind herself from Section 3 of the two different ways in which we obtained \mathfrak{T}_i from X . The first is the step-wise, ‘constructive’ way, starting from P_i and proceeding via the Ω_i s and the finsheaves $\mathbf{\Omega}_i$ thereof. The second is the ‘immediate’ way, via Papatriantafillou’s categorical results, going directly from X (now regarded not just as a *topological*, but as a *differential* manifold) and the CCDT \mathfrak{T}_∞ that it supports, to \mathfrak{T}_i , again starting from (i.e., with base topological spaces) Sorkin’s P_i s. Then one recalls from (17) that on these triads the vacuum Einstein equations of an *fcq*-version of Lorentzian vacuum Einstein gravity hold.
- Next, from Section 4 we recall that the said finitary differential triads comprise an inverse/direct system $\overset{=}{\mathcal{T}}$ possessing, following Sorkin via Papatriantafillou’s categorical perspective on ADG, the CCDT $\mathfrak{T}_\infty \equiv \mathbb{T}_\infty$ as a projective/inductive limit (18).
- Moreover, a plethora of finitary ADG-theoretic constructions, vital for the formulation of a finitary version of Lorentzian gravity regarded as a gauge theory, are based on those \mathfrak{T}_i s. These include for example the aforementioned *fcq*-vacuum Einstein equations, the *fcq-Einstein-Hilbert action functional* $\mathfrak{E}\mathfrak{H}_i$ from which these equations derive from variation with respect to the Lorentzian gravitational *fcq*-connections \mathcal{D}_i , and the *fcq-moduli spaces* $A_i(\mathcal{E}_i)/Aut\mathcal{E}_i$ of those gauge-equivalent (self-dual) *fcq*-spin-Lorentzian connections—as noted earlier, the gauge-theoretic configuration spaces of our *fcq*-version of Lorentzian (vacuum) Einstein gravity. Thus, it is fitting at this point to recall from Mallios and Raptis (2003)⁶³ the “*II-storeys’ tower of fcq-inverse and direct systems*” based on the \mathfrak{T}_i s in $\overset{=}{\mathcal{T}}$;⁶⁴ Papatriantafillou’s results secure that all these inverse-direct systems yield, like Sorkin’s original projective system $\overset{\leftarrow}{\mathcal{P}}$, their classical continuum counterparts at the limit of infinite resolution of the (base) P_i s. Equivalently, the continuum structures arise at the limit of infinite (topological)

⁶³ Expression (150) there.

⁶⁴ In the table below, the letter ‘v’ adjoined to our acronym ‘*fcq*’ stands for ‘(v)acuum’ (Mallios and Raptis, 2003). Also, the reader can refer to the latter paper (or of course to the ‘originals’ (Mallios, 1998a,b, 2001)) for the important notion of ‘*curvature space*,’ which however we will not be needing here.

‘Standing on the shoulders of triads’



U_i -refinement (Sorkin, 1991) (or equivalently, at the limit of infinite sheaf-theoretic localization of qausets—inhabiting the stalks of the respective finsheaves at the finitary level—over X ’s points).

- Of special interest to the proposed ‘resolution’ of the interior Schwarzschild singularity here, is the inverse system $\overleftarrow{\mathcal{E}}$ at level 5 in (25) above. *The projective limit of this system recovers the classical continuum vacuum Einstein equations over the whole (i.e., over all the points of) X* (19). In particular, we wish to emphasize that

the (vacuum) Einstein equations hold over the genuinely singular from the CDG-theoretic vantage point-mass m in the interior of X , and in no sense—at least in the differential geometric sense of the DGSs in which we are especially interested in the present paper—do they appear to break down there.

In this sense we say that the inner Schwarzschild singularity has been ‘resolved’ by finitary-algebraic ADG-theoretic means.

Below, we wish to make some further points in order to qualify more this ‘resolution’:

- First, as noted in Section 2, since in the ADG-theoretic perspective on GR it is the algebraic \mathbf{A} -connection \mathcal{D} and *not* the smooth metric g (as in the original formulation of the theory) that is the sole, fundamental (dynamical) variable, and since moreover ADG is genuinely smooth background manifold-independent, the usual conception of the inner Schwarzschild singularity as a DGS is *not* valid in our scheme since neither the metric nor its C^k -extensibility ($k = 0 \dots \omega$) are relevant, let alone important, issues in the theory.
- Related to the point above is the fact that in ADG we replace the usual CDG-based GR conception of a genuinely non-singular spacetime ‘*the solution metric holds (i.e., it is non-singular) in the entire manifold X* ’ by the expression that ‘*the field law (i.e., the differential equation of Einstein that \mathcal{D} defines via its curvature \mathcal{R}) is valid throughout all the field’s carrier (sheaf) space \mathcal{E} over the whole base topological space(time) X , which functional sheaf can in turn host all kinds of singularities.*’ Alias, there is no breakdown whatsoever of ‘differentiability,’ that is, the differential equation that \mathcal{D} defines, in our scheme. The ADG-gravitational field (\mathcal{D} , \mathcal{E}), and the dynamical differential equations that it defines via its curvature, $\mathcal{R}(\mathcal{D})(\mathcal{E}) = 0$, is not impeded at all by any singularities that the background topological space X (or the functional sheaf \mathcal{E} localized on it) might possess.
- One should note that the particular finitary-algebraic inner Schwarzschild singularity ‘resolution’ presented above is closely akin to (or one might even say that it follows suit from) the topological resolution of X *à la* Sorkin (Sorkin, 1991), in the following sense: as the ur-cell $\Lambda(m)|_{\mathcal{U}_i}$ blowing up and smearing the classically offensive point $m \in X$ becomes ‘smaller’ and ‘smaller’ with topological \mathcal{U}_i -refinement (resp., the topology τ_i generated by the open sets in the \mathcal{U}_i s becomes finer and finer), the law of gravity holds as close to the point-singularity m as one wishes to get (i.e., at every level ‘ i ’ of resolution or topological refinement of X by the open coverings \mathcal{U}_i). Furthermore, at the (projective) limit of infinite topological resolution

(refinement) of X into its points, one gets that (19) actually holds on (over) m itself.

- In connection with the last remarks, it is also worth pointing out that the law of gravity holds *both* at the ‘discrete,’ *fcq*-level of the P_i s ($\forall i$) and at the classical level limit corresponding to X , which further supports our *motto* that the ADG-picture of (vacuum) Lorentzian Einstein gravity (GR), and the *fcq*-version of it, is genuinely background-independent—i.e., whether that background is a ‘classical continuum’ or a ‘quantal discretum.’ *In toto*, this emphasizes that our ADG-perspective on gravity is manifestly (base) spacetime free (Mallios and Raptis, 2001, 2002, 2003, 2004). With respect to the CDG-problem of the inner Schwarzschild singularity and the usual divergence of the gravitational field strength (\mathcal{R}) in its vicinity, this freedom may be interpreted as follows: the vacuum Einstein equations hold *both* when a (locally) finite *and* an uncountable continuous infinity of degrees of freedom of the gravitational field are excited (as it were, when the gravitational field ‘occupies’ and effectuates a finite and an infinite number of point-events in the background space(time) X). Moreover, unlike the CDG-based picture of inner Schwarzschild singularity, no infinity at all (in the analytical sense of CDG)⁶⁵ for \mathcal{R} is involved as m is ‘approached’ (in the categorical limit sense of $\infty \leftarrow i$) by $\mathcal{R}_i(\mathcal{D}_i)$ upon (topological) refinement of the $\Lambda(m)|_{\mathcal{U}_i}$ s. There is no unphysical infinity associated with this ADG-picture of the inner Schwarzschild singularity, and in this sense the latter is ‘resolved into locally finite effects.’
- Of course, all this can be attributed to the fact that the base topological space(time) X (whether a continuum or a discontinuum) plays no role whatsoever in the inherently algebraic differential geometric mechanism of ADG, which, as noted earlier, derives from the algebra inhabited stalks of the (fin)sheaves involved and not from the base space which is merely a topological space. Technically speaking, this is reflected by the fact that the categorical in nature ADG-formulation of the relevant differential equations (here, the Einstein equations) involves (equations between) *sheaf morphisms*, and in particular, \mathbf{A} -morphisms such as \mathcal{R} . Sheaf morphisms by definition ‘see through’ the arbitrary base topological space X , which in turn serves only as a surrogate scaffolding, having no physical significance whatsoever as it plays no role in the gravitational dynamics—the (vacuum) Einstein differential equations (9). X is used only for the mathematical (:sheaf-theoretic) localization and concomitant gauging of the algebraic objects in the relevant (fin)sheaves (Mallios and Raptis, 2001).

⁶⁵For example, when m is relegated to X ’s boundary ∂X and a suitable topology is given to $\bar{X} = X \cup \partial X$, as $\mathcal{R} \rightarrow m$, \mathcal{R} diverges as $1/r^6$.

- Even more important than the remarks about the physical insignificance of the base space X , but closely related to them, is the issue of the \mathbf{A} -functoriality of dynamics already alluded to in Section 2. Namely, the fact that the vacuum Einstein equations (9) are (local) expressions of the curvature \mathcal{R} of the gravitational connection \mathcal{D} , which curvature is an \mathbf{A} -morphism (or \mathbf{A} -tensor)—a ‘geometrical object’ in ADG jargon (Mallios and Raptis, 2003), means that our generalized coordinates (or ‘measurements’) in the structure sheaf \mathbf{A} (that we assume to coordinatize the gravitational field \mathcal{D} and solder it on \mathcal{E} , which is anyway locally \mathbf{A}^n) respect the gravitational field (strength). Equivalently, it indicates that the field dynamics ‘sees through’ our (local) measurements in $\mathbf{A}(U)$. In turn, since all the singularities are inherent in \mathbf{A} —the structure sheaf of generalized algebras of ‘differentiable’ coordinate functions, it follows that the \mathbf{A} -functorial field dynamics ‘sees through’ the singularities built into the \mathbf{A} that we assume. Equivalently, but in a more philological sense, \mathbf{A} (and the singularities that it carries) is ‘transparent’ to the $\mathcal{R}(\mathcal{D})$ engaging into the gravitational field dynamics—the differential equations of Einstein (9). *In summa*, the field $(\mathcal{E}, \mathcal{D})$ (and the differential equation that it defines via its curvature) does not stumble on or break down at any singularity inherent in \mathbf{A} , since it passes ‘through’ (or over) them. In this sense, the term ‘singularity-resolution’ is not a very accurate name to describe how ADG evades singularities. Perhaps a better term is ‘*dissolution*’ or ‘*absorption*’ in \mathbf{A} .

A good example of the aforesaid singularity-dissolution or absorption in \mathbf{A} is the ADG-theoretic evasion of the inner Schwarzschild singularity regarded now as a time-extended (distributional) *spacetime foam dense singularity* in the sense of Mallios and Rosinger (Mallios and Rosinger, 1999, 2001; Mallios, 2001, 2002; Mallios and Rosinger, 2002). We briefly discuss this ‘dissolution’ next, leaving a thorough treatment to the forthcoming ‘paper-book’ (Mallios and Raptis, 2004).

5.4. A Second ‘Resolution’ of the Inner Schwarzschild Singularity via Spacetime Foam Dense Singularities: A ‘Temporal,’ Distributional Time-Line Resolution

There is another possible evasion of the interior Schwarzschild singularity by ADG-theoretic and finitary means, by regarding it this time not as a ‘static’ (stationary), ‘spatial,’ point-localized singularity as above, but as an extended, distributional one (much in the sense of SFSs above) extending along the ‘wristwatch’ (locally) Euclidean time-axis L_t of the point-particle (20).

The idea is to regard L_t as being inhabited by so-called ‘*spacetime foam dense singularities*’ à la (Mallios and Rosinger, 2001, 2002). On the side, in

mathematics these are singularities of generalized functions (distributions)—situated on dense subsets of finite-dimensional Euclidean and locally Euclidean space(time)s (manifolds)—functions which have been used as coefficients in and have been occurring as solutions of non-linear (both hyperbolic and elliptic) partial differential equations, as originally discovered and subsequently developed entirely algebraically by Rosinger in a series of papers (Rosinger, 1990, 1999,?, 2005). *En passant*, these distributions can be organized into differential algebras generalizing (and including) both the usual smooth functions $C^\infty(M)$ on manifolds and the well known linear distributions of Schwartz. They form the basis of Rosinger’s non-linear distribution theory. In physics, interest in such singularities has arisen recently in the study of ‘spacetime foam’ structures in GR and QG, as studied primarily by the Polish school of Heller *et al.* in the context of generalized differential spaces (Heller, 1991, 1993; Heller *et al.*, 1989; Heller and Sasin, 1995b; Gruszczak and Heller, 1993; Heller, 1992; Heller and Sasin, 1995a).

In the context of (applications of) ADG, the said algebras have been organized into sheaves and used as structure sheaves in the theory, replacing and generalizing (actually, containing!) the classical one C_X^∞ . Indeed, classical (CDG) constructions and results, normally based on C_M^∞ over a differential manifold M , (e.g., de Rham’s theorem, Poincaré’s lemma, de Rham cohomology, Weil’s integrality theorem, the Chern-Weil theorem *etc*), carry through, virtually unaltered, to the ‘ultra-singular’ realm of the spacetime foam dense singularities of the said generalized functions (Mallios and Rosinger, 2001, 2002; Mallios, 2002); moreover, the vacuum Einstein equations are seen again to hold, in full force, in their very presence (Mallios, 2001, 2002).

To comment a bit on the dense singularities, they are arguably the most robust and numerous singularities that have appeared so far in the general theory of non-linear partial differential equations, but three of their most prominent features that we would like to highlight here, in comparison to the usual singularities carried by C_X^∞ , are:

- First, their cardinality. These are singularities on arbitrary subsets of the underlying topological space(time) X . In particular, they can be concentrated on *dense* subsets of X , under the proviso that their complements, consisting of non-singular (regular) points, are also dense in X . In case X is a Euclidean space or a finite-dimensional manifold, *the cardinality of the set of singular points may be larger than that of the regular ones*. For instance, when one takes $X = \mathbb{R}$ (as we intuit to do here with L_t), the dense singular subsets of it may have the cardinal of the continuum—i.e., the singularities are situated on the irrational numbers, while the regular ones are also dense but countable in \mathbb{R} and situated, say, on the rationales.
- Second, their situation in the manifold’s *bulk*. As it is evident from the above, the dense singularities, apart from their uncountable multiplicity,

are not situated merely at the boundary of the underlying (topological) space(time) (manifold), but occupy ‘central’ points in its ‘bulk-interior.’ This is in striking contrast to the usual theory of C^∞ -smooth spacetime singularities that we briefly revisited in Section 2 (Hawking and Ellis, 1973; Clarke, 1986, 1993; Rendall, 2005), which we may thus coin ‘separated and isolated,’ or ‘solitary,’ or even ‘*spacetime marginal*’ for effect. This situation is also in contrast to the ‘algebraically generalized differential spaces’ (:spacetime foam) approach to GR and QG of Heller *et al.*, as they too assume (even though they too tend to employ sheaf-theoretic methods) that singularities—in fact, merely *nowhere dense singularities* (not dense ones!) in the sense of Mallios and Rosinger (1999)—sit right at the edge of the spacetime manifold (see Heller (1991, 1992, 1993); Heller and Sasin (1993, 1995a,b); Heller *et al.* (1989), and especially Gruszczak and Heller (1993)).

- And third, as briefly alluded to above, the differential algebras of generalized functions in Rosinger’s non-linear distribution theory contain both the usual algebra $C^\infty(X)$ of smooth functions and Schwartz’s linear distributions (Mallios and Rosinger, 2001, 2002). Furthermore, these non-linear distributions, either with nowhere dense, or even more prominently, with dense singularities, have proven to be more versatile (and potentially more useful in differential geometric applications) than the, quite popular lately in the theory of non-linear PDEs, non-linear Colombeau distributions (Colombeau, 1984).⁶⁶

5.5. Two Alternative Distributional ADG-Resolutions of the Inner Schwarzschild Singularity

Like in the point-resolution presented above, here too we can evade by ADG-theoretic means the interior Schwarzschild singularity, regarded as an extended distributional (:SFS-like) spacetime foam dense singularity along $L_t \simeq \mathbb{R}$, in two different ways—one ‘direct,’ the other ‘indirect’ and along the ‘finitary’ lines of Sorkin. Let us briefly mention the two strategies, leaving the rather lengthy technical details for Mallios and Raptis (2004).

- **‘Direct’ distributional evasion:** Here, following Mallios (2001, 2002), we can directly employ sheaves of Rosinger’s generalized functions hosting dense singularities on L_t as coordinate structure sheaves in the theory. Then, we straightforwardly borrow the main result from Mallios (2001, 2002), namely, that Einstein’s equations hold over all L_t when Rosinger’s spacetime foam sheaves are used as **A**. We call this strategy ‘direct,’

⁶⁶ See Mallios and Rosinger (2001) for a discussion of the (differential geometric) virtues of Rosinger’s distributions compared to Colombeau’s.

because, like in the first ‘non-constructive’ point-resolution before which evoked Papatiantafillou’s results and straightforwardly defined finitary differential triads on the $\overset{U_i}{\sim}$ -moduli spaces P_i , one can directly define spacetime foam differential triads without having to go ‘constructively,’ in a roundabout way, via finitary coverings, finsheaves (of incidence algebras) *etc.* The latter we can accomplish in the second possible strategy briefly described next.

- **‘Indirect’ distributional evasion:** Here, we combine the approach of Mallios-Rosinger in Mallios and Rosinger (2001, 2002) with Sorkin’s in Sorkin (1991) and let $X \equiv I \subset L_t \simeq \mathbb{R}$ (I a bounded interval of the real line as befits a physically realistic point-particle of finite lifetime) be covered by locally finite ‘singularity-open coverings.’ These are covering families of open subsets of X containing singularities (of Rosinger’s generalized functions) densely at their points. Then we go to finsheaves (of incidence algebras) and the finitary differential triads picture thereof so as to show that for each such covering the vacuum Einstein equations hold *à la* (Mallios and Raptis, 2003) and (Mallios and Raptis, 2003), and finally we pass to the classical ‘continuum’ projective limit of maximum topological-*cum*-singularity refinement to show that the vacuum Einstein equations hold over the whole (space)time—in particular, over all X . To be precise, and in keeping with Sorkin’s inverse limit result mentioned earlier, the vacuum Einstein equations can be seen to hold over all the densely singular points of X —itself assumed to be populated with spacetime foam dense singularities—when recovered as a dense subset of (closed points of) the non-Hausdorff inverse limit space of Sorkin’s finitary substitutes and the differential triads they support relative to the said locally finite open singularity-covers.

6. EPILEGOMENA: IMPLICATIONS FOR QUANTUM GRAVITY

In this concluding section we remark briefly on two issues. First, how ADG may prove to be a suitable theoretical framework in which to formulate a genuinely background-independent QG. Also, since ADG appears to evade completely (gravitational) singularities (Mallios, 2001, 2003, 2002, 2005a,b; Mallios and Raptis, 2004), we touch in its light on the nowadays general consensus (or at least, the wishful expectation) that QG should resolve, or ultimately remove, spacetime singularities (Hawking and Penrose, 1996; Penrose, 2003; Husain and Winkler, 2004). In this context, we loosely compare the evasion of the interior Schwarzschild singularity presented above with a similar resolution of it achieved very recently by the methods and results of Loop QG (LQG) in Modesto (2004),

making some relevant comments in the process. However, we leave a thorough discussion of what follows to Mallios and Raptis (2004).

- *Genuinely background-independent QG.* A major issue in QG, especially in non-perturbative canonical QGR (Thiemann, 2001) in its connection based LQG version (Rovelli and Smolin, 1990; Thiemann, 2002; Smolin, 2004), is to formulate the theory in a genuinely background-independent fashion (Alvarez, 2004). In a nutshell, by ‘*background independence*’ it is meant ‘*background metric independence*.’ That is, unlike in the usual (mainly perturbative) approaches to QG where one fixes a (usually flat, Minkowski) background metric in order to formulate the quantum dynamics (and expand the relevant quantities about it, as well as to impose physically meaningful commutation relations among them), here there is no such desire since, anyway, it appears to be begging the question to fix *a priori* (and by hand!), and moreover to duplicate, the supposedly sole dynamical variable of GR—the spacetime geometry (metric). Ashtekar and coworkers have succeeded over the years in formulating LQG in a manifestly fixed background metric independent way (Ashtekar and Lewandowski, 2004). Alas, *a smooth spacetime manifold is still retained in the background* (Ashtekar, 2003)—or else, how could one still use differential geometric ideas and constructions (Ashtekar and Lewandowski, 1995) in QG research? For example, as noted earlier, the new connection variables (Ashtekar, 1986) employed in LQG are *smooth* (spin-Lorentzian) connections based on a differential spacetime manifold, let alone that the smooth metric is still implicitly present in the guise of the smooth comoving tetrad (:*vierbein*) field variables (first-order formalism). All this is another instance of the aforementioned base manifold and CDG-conservatism and monopoly.

By contrast, in ADG GR is not only formulated, as befits a purely gauge theory, solely in terms of the gravitational **A**-connection variable without at all the presence of a metric (‘half-order formalism’), but also, *a fortiori, no base differential spacetime manifold appears at all in the theory.* In this sense, the ADG-approach to gravity—classical or quantum—is genuinely background-independent (Mallios and Raptis, 2003, 2004). On the other hand, it is plain that since singularities are inherent in C_M^∞ (i.e., in the background differential spacetime manifold M), loop QG still has to reckon with them—that is, they are still problems for the theory and thus the theory still aims at resolving them somehow. We thus comment on a recent resolution of the inner Schwarzschild singularity by LQG means (Modesto, 2004) in the next paragraph, comparing it at the same time with ours above.

- *Comparison with a recent resolution of the inner Schwarzschild singularity by LQG methods.* As noted before, there is currently optimism among

theoretical physicists that QG will shed more light and ultimately (re)solve the problem of smooth spacetime singularities in GR. Notably, within the past three years, in the context of Loop Quantum Cosmology it has been shown that the initial ('Big Bang') singularity predicted by GR can be indeed resolved (Bojowald, 2001a,b; Ashtekar *et al.*, 2003; Husain and Winkler, 2004). However, even more remarkable for the present paper is the following very recent result of (Modesto, 2004), which was also arrived at by LQG means: In one sentence, *the Schwarzschild black hole singularity of the classical theory (GR) 'disappears' in QG*. In this penultimate paragraph we would like to describe briefly this 'disappearance,' comment on it and juxtapose it with the 'resolution' of the same singularity that we achieved herein by ADG-theoretic means.

Let us first note that since, as mentioned before, LQG, although background metric independent, still employs a base differential (spacetime) manifold for its constructions, the problem of singularities in the classical theory persists and has to be reckoned with in the quantum theory. In this regard, it is fair to say that loop QG 'expects' that the 'true' quantum theory of gravity it aspires to be should ultimately resolve or remove the singularities and the associated pathological infinities of the classical theory (Husain and Winkler, 2004). Briefly, in Modesto (2004) the interior Schwarzschild singularity appears to be resolved as follows:⁶⁷

1. To begin with, one expresses the Ricci scalar curvature, which as noted earlier blows up as $1/r^6$ near the interior ($r = 0$) Schwarzschild singularity, in terms of the spacetime volume.
2. Then, one evokes *the* major result in LQG, namely, that the said volume is quantized—i.e., it is promoted to a volume operator having a discrete eigen-spectrum. Parenthetically, we mention that this volume-quantization (Ashtekar and Lewandowski, 1997b) is just one of a series of significant results in Ashtekar's quantum (Riemannian) geometry programme accompanying LQG (Ashtekar, 2003; Thiemann, 2001, 2002; Smolin, 2004), along with the quantization of length (Thiemann, 1998) and area (Ashtekar and Lewandowski, 1997a) (see also Rovelli and Smolin (1995); Rovelli (2002)). Thus, near the Schwarzschild black hole, \mathcal{R} is rendered finite and the classical infinities are controlled ('regularized') by quantum theory.
3. Moreover, one can show that the said 'regularization' is not 'kinematical' and without physical significance—one that is *a priori* fixed by hand like for example the space(time) discretizations in lattice QCD—but it is a dynamical one. This is so because the Hamiltonian (constraint), which regulates the dynamical time-evolution in the canonical approach

⁶⁷ The reader is referred to Modesto (2004) for detailed arguments, calculations and pertinent citations.

to GR classically underlying LQG (Thiemann, 2001, 2002), can also be expressed in terms of the volume operator. Thus, as Modesto shows, *the spacetime can be dynamically extended past the interior Schwarzschild singularity, with no infinity involved at all.*

4. On the other hand, from a *differential* geometric viewpoint, the upshot of all this is that the said dynamical evolution, which is classically represented by a *differential* equation on the spacetime continuum,⁶⁸ is now substituted, in view of the said quantization of spacetime geometry in LQG, by a ‘discrete,’ *difference equation* (discretely parameterized by the coefficients of the physical quantum eigenstates of the volume operator—in a quantum cosmological setting, see also Bojowald (2001b)). *In summa*, one can say that *the inner Schwarzschild singularity is resolved due to the quantization of spacetime itself.*

Based on the brief description above, our comparison of the two ‘resolutions’ of the interior Schwarzschild singularity focuses on two fundamental in our opinion differences:

- I. Unlike in the LQG ‘resolution’ where a quantization of spacetime appears to be necessary, in the ADG ‘resolution’ this is not so, for the theory is ‘intrinsically background spacetimeless.’ That is, the theory is indifferent as to whether that background is a ‘classical continuum’ or a ‘quantal discretum,’ since the dynamical Einstein equations hold both at the ‘discrete-quantal’ (finitary) level *and* at the ‘continuous-classical’ (infinitary) one (Mallios and Raptis, 2003, 2004). In the ADG perspective on gravity, where the sole dynamical variable is an algebraic connection field \mathcal{D} on a vector/algebra sheaf \mathcal{E} (on an in principle arbitrary topological space X), the quest for a quantization of spacetime is virtually begging the question: *in the first place, in ADG, what ‘spacetime’ is one talking about?*. Another way to say this is that, from the ADG-viewpoint, gravity (i.e., the dynamically autonomous gravitational field \mathcal{D} defining the Einstein equations via its curvature) has nothing to do with a background ‘space(time)’ (in our case, the background X which serves only as a surrogate topological space for the sheaf-theoretic localization and representation of the relevant sheaves; e.g., \mathbf{A} , \mathcal{E} , and \mathcal{D} acting on it), so that a possible quantum theory of the former is in no need of a quantum description of the latter (Mallios and Raptis, 2002, 2003, 2004). As a consequence of this difference.
- II. Unlike in the loop QG ‘resolution’ where the said spacetime quantization and concomitant discretization appears to necessitate the abandoning of

⁶⁸ After all, the Hamiltonian (constraint) in the classical canonical theory (GR) is the generator of time-diffeomorphisms.

the picture of ‘gravitational dynamical evolution’ as a differential equation proper (and, as a result, the abandonment of differential geometric ideas in the quantum regime—e.g., see Isham quotation from Isham (1991) before), in the ADG ‘resolution’ all the differential geometric machinery (of the background spacetime continuum) is retained in full effect (Mallios and Raptis, 2002, 2003, 2004). Moreover, this is so manifestly independently of that background, and *a fortiori*, even if that background is taken to be a ‘discretum’ where differential geometric ideas would traditionally—i.e., from the CDG-viewpoint of the continuous manifold—be expected to fail to apply.

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REFERENCES

- Álvarez, E. (2004). *Quantum gravity*, pre-print [gr-qc/0405107].
- Ashtekar, A. (1986). New variables for classical and quantum gravity. *Physical Review Letters* **57**, 2244.
- Ashtekar, A. (2003). *Quantum geometry and its ramifications*. In *The Future of Theoretical Physics and Cosmology: Celebrating Stephen Hawking’s 60th Birthday*, G. W. Gibbons, E. P. S. Shellard, and S. J. Rankin, eds., Cambridge University Press, Cambridge.
- Ashtekar, A., Bojowald, M., and Lewandowski, J. (2003). *Mathematical Structure of Loop Quantum Cosmology*, pre-print [gr-qc/0304074].
- Ashtekar, A. and Lewandowski, J. (1995). Differential geometry on the space of connections via graphs and projective limits. *Journal of Geometry and Physics* **17**, 191.
- Ashtekar, A. and Lewandowski, J. (1997a). Quantum theory of geometry I: Area operators. *Classical and Quantum Gravity* **14**, A55.
- Ashtekar, A. and Lewandowski, J. (1997b). Quantum theory of geometry II: Volume operators. *Advances in Theoretical and Mathematical Physics* **1**, 388.
- Ashtekar, A. and Lewandowski, J. (2004). *Background independent quantum gravity: A status report*, pre-print [gr-qc/0404018].

- Bergmann, P. G. (1979). *Unitary field theory: Geometrization of physics or physicalization of geometry?*. In *The 1979 Berlin Einstein Symposium*, Lecture Notes in Physics, Springer-Verlag, Berlin Heidelberg New York.
- Bojowald, M. (2001a). Absence of singularity in loop quantum cosmology. *Physical Review Letters* **86**, 5227.
- Bojowald, M. (2001b). Loop quantum cosmology IV: Discrete time evolution. *Classical and Quantum Gravity* **18**, 1071.
- Bombelli, L., Lee, J., Meyer, D., and Sorkin, R. D. (1987). Space-time as a causal set. *Physical Review Letters* **59**, 521.
- Clarke, C. J. S. (1986). *Singularities: Local and global aspects*. In *Topological Properties and Global Structure of Space-Time*, P. G. Bergmann and V. De Sabbata, eds., NATO ASI Series, Plenum Press, New York and London.
- Clarke, C. J. S. (1993). *The Analysis of Space-Time Singularities*, Cambridge Lecture Notes in Physics, Cambridge University Press, Cambridge.
- Colombeau, J.-F. (1984). *New Generalized Functions and Multiplication of Distributions*, Mathematical Studies **84**, North-Holland, Amsterdam.
- Einstein, A. (1956). *The Meaning of Relativity*, 5th edn., Princeton University Press, Princeton.
- Finkelstein, D. (1958). Past-future asymmetry of the gravitational field of a point particle. *Physical Review* **110**, 965.
- Geroch, R. (1968). What is a singularity in General Relativity? *Annals of Physics* **48**, 526.
- Geroch, R., Kronheimer, E. H., and Penrose, R. (1972). Ideal points in space-time and singularities. *Proceedings of the Royal Society London A* **327**, 545.
- Grauert, H. and Remmert, R. (1984). *Coherent Analytic Sheaves*, Springer-Verlag, Berlin.
- Gruszczak, J. and Heller, M. (1993). Differential structure of space-time and its prolongations to singular boundaries. *International Journal of Theoretical Physics* **32**, 625.
- Hawking, S. W. and Ellis, G. F. R. (1973). *The Large Scale Structure of Space-Time*, Cambridge University Press, Cambridge.
- Hawking, S. W. and Penrose, R. (1970). The singularities of gravitational collapse and cosmology. *Proceedings of the Royal Society London A* **314**, 529.
- Hawking, S. W. and Penrose, R. (1996). *The Nature of Space and Time*, Princeton University Press, Princeton.
- Heller, M. (1991). Algebraic foundations of the theory of differential spaces. *Demonstratio Mathematicum* **24**, 349.
- Heller, M. (1992). Einstein algebras and general relativity. *International Journal of Theoretical Physics* **31**, 277.
- Heller, M. (1993). Geometry of transition to quantum gravity regime. *Acta Physica Polonica B* **24**, 911.
- Heller, M., Multarzynski, P., and Sasin, W. (1989). The algebraic approach to space-time geometry. *Acta Cosmologica* **XVI**, 53.
- Heller, M. and Sasin, W. (1993). Generalized Friedman's equation and its singularities. *Acta Cosmologica* **XIX**, 23.
- Heller, M. and Sasin, W. (1995a). Sheaves of Einstein algebras. *International Journal of Theoretical Physics* **34**, 387.
- Heller, M. and Sasin, W. (1995b). Structured spaces and their application to relativistic physics. *Journal of Mathematical Physics* **36**, 3644.
- Husain, V. and Winkler, O. (2004). *Quantum resolution of black hole singularities*, pre-print [gr-qc/0410125].
- Isham, C. J. (1991). *Canonical groups and the quantization of geometry and topology*. In *Conceptual Problems of Quantum Gravity*, A. Ashtekar and J. Stachel, eds., Birkhäuser, Basel.
- Kopperman, R. D. and Wilson, R. G. (1997). Finite approximation of compact Hausdorff spaces. *Topology Proceedings* **22**, 175.

- Kruskal, M. (1960). Maximal extension of the Schwarzschild metric. *Physical Review* **119**, 1743.
- Mallios, A. (1986). *Topological Algebras. Selected Topics*, North-Holland, Amsterdam.
- Mallios, A. (1988). *On the existence of \mathcal{A} -connections*. *Abstracts of the American Mathematical Society* **9**, 509.
- Mallios, A. (1992). On an abstract form of Weil's integrality theorem. *Note di Matematica* **12**, 167 (invited paper).
- Mallios, A. (1993). The de Rham-Kähler complex of the Gel'fand sheaf of a topological algebra. *Journal of Mathematical Analysis and Applications* **175**, 143.
- Mallios, A. (1998a). *Geometry of Vector Sheaves: An Axiomatic Approach to Differential Geometry*, vols. 1–2, Kluwer, Dordrecht.
- Mallios, A. (1998b). On an axiomatic treatment of differential geometry via vector sheaves. Applications. *Mathematica Japonica (International Plaza)* **48**, 93 (invited paper).
- Mallios, A. (2001). Abstract differential geometry, general relativity and singularities. In *Unsolved Problems in Mathematics for the 21st Century: A Tribute to Kiyoshi Iséki's 80th Birthday*, J. M. Abe and S. Tanaka, eds., 77, IOS Press, Amsterdam (invited paper).
- Mallios, A. (2002). Abstract differential geometry, singularities and physical applications. In *Topological Algebras with Applications to Differential Geometry and Mathematical Physics*, P. Strantzalos and M. Fragouloupoulou, eds., Proceedings of a Fest-Colloquium in Honour of Professor Anastasios Mallios (16–18/9/1999), Department of Mathematics, University of Athens Publications.
- Mallios, A. (2003). *Remarks on "singularities."* In *Progress in Mathematical Physics*, F. Columbus, ed., Nova Science Publishers, Hauppauge, New York (invited paper) [gr-qc/0202028] (in press).
- Mallios, A. (2004a). On localizing topological algebras. *Contemporary Mathematics* **341**, 79 [gr-qc/0211032].
- Mallios, A. (2004b). *Geometry and physics of today*, pre-print (2004) [physics/0405112].
- Mallios, A. (2005a). *Quantum gravity and "singularities,"* Note di Matematica (invited paper) [physics/0405111] (in press).
- Mallios, A. (2005). *Modern Differential Geometry in Gauge Theories. Vol. I: Maxwell Fields, Vol. II: Yang-Mills Fields*. Birkhauser, Boston.
- Mallios, A. and Raptis, I. (2001). Finitary spacetime sheaves of quantum causal sets: Curving quantum causality. *International Journal of Theoretical Physics* **40**, 1885 [gr-qc/0102097].
- Mallios, A. and Raptis, I. (2002). Finitary Čech-de Rham Cohomology. *International Journal of Theoretical Physics* **41**, 1857 [gr-qc/0110033].
- Mallios, A. and Raptis, I. (2003). Finitary, causal and quantal vacuum Einstein gravity. *International Journal of Theoretical Physics* **42**, 1479 [gr-qc/0209048].
- Mallios, A. and Raptis, I. (2004). C^∞ -Smooth singularities exposed: Chimeras of the differential spacetime manifold, 'paper-book'/research monograph [gr-qc/0411121].
- Mallios, A. and Rosinger, E. E. (1999). Abstract differential geometry, differential algebras of generalized functions and de Rham cohomology. *Acta Applicandae Mathematicae* **55**, 231.
- Mallios, A. and Rosinger, E. E. (2001). Space-time foam dense singularities and de Rham cohomology. *Acta Applicandae Mathematicae* **67**, 59.
- Mallios, A. and Rosinger, E. E. (2002). Dense singularities and de Rham cohomology. In *Topological Algebras with Applications to Differential Geometry and Mathematical Physics*, P. Strantzalos and M. Fragouloupoulou, eds., Proceedings of a Fest-Colloquium in Honour of Professor Anastasios Mallios (16–18/9/1999), Department of Mathematics, University of Athens Publications.
- Modesto, L. (2004). *Disappearance of the black hole singularity in quantum gravity*, pre-print [gr-qc/0407097].
- Papatriantafillou, M. H. (2000). The category of differential triads. In *Proceedings of the 4th Panhellenic Conference on Geometry (Patras, 1999)*, *Bulletin of the Greek Mathematical Society* **44**, 129.

- Papatriantafillou, M. H. (2001). Projective and inductive limits of differential triads. In *Steps in Differential Geometry, Proceedings of the Institute of Mathematics and Informatics Debrecen (Hungary)*, 251.
- Papatriantafillou, M. H. (2003). On a universal property of differential triads, pre-print [An earlier draft of this paper, which is the one we possess, was titled *Morphisms of Differential Triads*].
- Papatriantafillou, M. H. (2004). Initial and final differential structures. In *Proceedings of the International Conference on Topological Algebras and Applications: 'Non-normed Topological Algebras'*, Rabat, Maroc (2003), ENST publications **2**, 115.
- Papatriantafillou, M. H. (2005). *Abstract Differential Geometry. A Categorical Perspective* (in preparation).
- Penrose, R. (2003). The problem of spacetime singularities: Implications for quantum gravity? In *The Future of Theoretical Physics and Cosmology: Celebrating Stephen Hawking's 60th Birthday*, G. W. Gibbons, E. P. S. Shellard, and S. J. Rankin, eds., Cambridge University Press, Cambridge.
- Raptis, I. (2000a). Algebraic quantization of causal sets. *International Journal of Theoretical Physics* **39**, 1233 [gr-qc/9906103].
- Raptis, I. (2000b). Finitary spacetime sheaves. *International Journal of Theoretical Physics* **39**, 1703 [gr-qc/0102108].
- Raptis, I. (2003). Quantum space-time as a quantum causal set. In a significantly modified and expanded version of the Los Alamos e-arXiv posted paper in the volume *Progress in Mathematical Physics*, F. Columbus, ed., Nova Science Publishers, Hauppauge, New York (invited paper) [gr-qc/0201004].
- Raptis, I. and Zapatrin, R. R. (2000). Quantization of discretized spacetimes and the correspondence principle. *International Journal of Theoretical Physics* **39**, 1 [gr-qc/9904079].
- Raptis, I. and Zapatrin, R. R. (2001). Algebraic description of spacetime foam. *Classical and Quantum Gravity* **20**, 4187 [gr-qc/0102048].
- Rendall, A. D. (2005). *The nature of spacetime singularities*, pre-print [gr-qc/0503112].
- Rosinger, E. E. (1990). *Non-Linear Partial Differential Equations. An Algebraic View of Generalized Solutions*, North-Holland, Amsterdam.
- Rosinger, E. E. (1999). *Space-time foam differential algebras of generalized functions and a global Cauchy-Kovalevskaya theorem*, Technical Report UPWT 99/8, Department of Mathematics, University of Pretoria, Republic of South Africa.
- Rosinger, E. E. (1999). *Differential algebras with dense singularities on manifolds*, Technical Report UPWT 99/9, Department of Mathematics, University of Pretoria, Republic of South Africa.
- Rosinger, E. E. (2005). *Dense Singularities and Non-Linear Partial Differential Equations*, monograph (in press).
- Rovelli, C. (2002). Partial observables. *Physical Review D* **65**, 124013 [gr-qc/0110035].
- Rovelli, C. and Smolin, L. (1990). Loop space representation of quantum general relativity. *Nuclear Physics B* **331**, 80.
- Rovelli, C. and Smolin, L. (1995). Discreteness of area and volume in quantum gravity. *Nuclear Physics B* **442**, 593. [Erratum: *Nuclear Physics B* **456**, 734.]
- Schmidt, B. (1971). A new definition of singular points in general relativity. *General Relativity and Gravitation* **1**, 269.
- Smolin, L. (2004). *An invitation to loop quantum gravity*, pre-print [gr-qc/0408048].
- Sorkin, R. D. (1990). Does a discrete order underlie spacetime and its metric? In *Proceedings of the Third Canadian Conference on General Relativity and Relativistic Astrophysics*, F. Cooperstock and B. Tupper, eds., World Scientific, Singapore.
- Sorkin, R. D. (1991). Finitary substitute for continuous topology. *International Journal of Theoretical Physics* **30**, 923.
- Sorkin, R. D. (1995). A specimen of theory construction from quantum gravity. In *The Creation of Ideas in Physics*, J. Leplin, ed., Kluwer Academic Publishers, Dordrecht [gr-qc/9511063].

- Sorkin, R. D. (1997). Forks in the road, on the way to quantum gravity. *International Journal of Theoretical Physics* **36**, 2759 [gr-qc/9706002].
- Thiemann, T. (1998). A length operator for canonical quantum gravity. *Journal of Mathematical Physics* **39**, 3372.
- Thiemann, T. (2001). *Introduction to modern canonical quantum general relativity*, pre-print [gr-qc/0110034].
- Thiemann, T. (2002). *Lectures on Loop Quantum Gravity*, pre-print [gr-qc/0210094].
- Thompson, G. (2001). *On Leibniz*, Wadsworth Philosophical Series, Wadsworth, USA.
- Vassiliou, E. (1994). On Mallios' \mathcal{A} -connections as connections on principal sheaves. *Note di Matematica* **14**, 237.
- Vassiliou, E. (1999). Connections on principal sheaves. In *New Developments in Differential Geometry*, J. Szenthe, ed., Kluwer Academic Publishers, Dordrecht.
- Vassiliou, E. (2000). On the geometry of associated sheaves. *Bulletin of the Greek Mathematical Society* **44**, 157.
- Vassiliou, E. (2005). *Geometry of Principal Sheaves*, Springer-Verlag, Berlin-New York.
- Zapatrin, R. R. (1998). Finitary algebraic superspace. *International Journal of Theoretical Physics* **37**, 799.
- Zapatrin, R. R. (2001). *Continuous limits of discrete differential manifolds*, pre-print. [This pre-print can be retrieved from Roman Zapatrin's personal webpage: www.isiosf.isi.it/~zapatrin.]
- Zapatrin, R. R. (2002). *Incidence algebras of simplicial complexes*, *Pure Mathematics and its Applications* (in press) [math.CO/0001065].